Equilibrium potentials and Liouville random measures for Dirichlet forms

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Introduction

$(\mathcal{E}, \mathcal{F})$: strongly local Dirichlet form on $L^2(E; m)$, $(E = \mathbb{R}^2)$
$\mathbb{M} = (X_t, \mathbb{P}_{\mathbf{x}})$: associated diffusion process

$\mathbb{M}$ is transient $\Rightarrow \mathbb{P}_{\mathbf{x}}(\sigma_B < \infty) = R\mu_B$, $\exists \mu_B(\overline{B}) = \mathrm{Cap}(B)$

μ_B:equilibrium measure
([M.Fukushima, Y.Oshima and M.Takeda; de Gruyter, 2010])
$\mu^B = \mu_B/\mathrm{Cap}(B)$: normalized equilibrium measure
$R\mu^B = \frac{1}{\mathrm{Cap}(B)} = f(B)$ q.e. on $\overline{B}$: Robin constant

M is the 2-dimensional Brownian motion, $B = B(\mathbf{x}, r) \Rightarrow$

$$U\nu^{\mathbf{x},r}(\mathbf{y}) = \frac{1}{\pi}\int \log \frac{1}{|\mathbf{y}-\mathbf{z}|}\nu^{\mathbf{x},r}(d\mathbf{z}) = \frac{1}{\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}| \vee r}$$

$$\left(= \frac{1}{\pi}\log\frac{1}{r} \quad \text{on } B(\mathbf{x}, r)\right)$$

$\nu^{\mathbf{x},r}$: uniform probability measure on $\partial B(\mathbf{x}, r)$: equilibrium measure
$\frac{1}{\pi}\log\frac{1}{r}$: Robin constant relative to logarithmic potential
([S.C.Port and C.J.Stone, Academic Press, 1978])

Purpose of this talk

$E = \mathbb{R}^2$ or bounded domain of $\mathbb{R}^2$
m: Lebesgue measure on E
$(\mathcal{E}, \mathcal{F})$: strongly local Dirichlet form on $L^2(E; m)$ defined by

$$\mathcal{E}(u, v) = \sum_{i,j=1}^{2} \int_E a_{ij}(\mathbf{x}) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} d\mathbf{x} \quad u, v \in \mathcal{F} = H_0^1(E)$$

$(a_{ij}(\mathbf{x}))$: uniformly elliptic family of bounded symmetric measurable functions.
$\mathbb{M} = (X_t, \mathbb{P}_\mathbf{x})$: associated diffusion process

Purpose of this talk:

- To give the equilibrium measures with explicit expression
- Use it to construct the Liouville random measures.

Assume that $E = \mathbb{R}^2$

Then the following result holds:

- The transition function $p_t(x, y)$ has (t, x, y)-continuous version satisfying Gaussian estimate.
- The Harnack inequality holds: If $u(\mathbf{x})$ is harmonic on a domain D, then for any disk with $\overline{B} \subset D$, there exists a constant C such that

$$\max\{u(\mathbf{x}) : \mathbf{x} \in \overline{B(\mathbf{x}_0, r)}\} \leq C \min\{u(\mathbf{x}) : \mathbf{x} \in \overline{B(\mathbf{x}_0, r)}\}$$

Since the resolvent density $r_\alpha(\mathbf{x}, \mathbf{y})$ is bounded from below by a positive constant on any compact set, any compact set of positive m-measure is an admissible set.

Potential of perturbed process

Take $F = \overline{B(S+1)} \setminus B(S)$ as an admissible set and put $g(\mathbf{x}) = 1_F(\mathbf{x})$
$\mathbb{M}^g$ is the diffusion process corresponding to
$\mathcal{E}^g(u,v) = \mathcal{E}(u,v) + (u,v)_{L^2(g\cdot m)}$

R^g be the potential of $\mathbb{M}^g$:

$$R^g f(\mathbf{x}) = \mathbb{E}_{\mathbf{x}}\left(\int_0^\infty e^{-A_t} f(X_t)dt\right), \quad A_t = \int_0^t g(X_s)ds.$$

$\exists r^g(\mathbf{x},\mathbf{y})$: $(\mathbf{x},\mathbf{y})$-continuous (on $B(S)\times B(S)\setminus \mathbf{d}$) density of $R^g(\mathbf{x}, d\mathbf{y})$

$\mathcal{S}_0^{g,(0)}$: family of measures with finite 0-order energy integral
$\mathcal{M}_0 = \{\mu = \mu_1 - \mu_2 : \mu_i \in \mathcal{S}_0^{g,(0)}, \mu_i(E) < \infty\}$.

$\check{R}_\alpha$: resolvent of the time changed process of $\mathbb{M}$ by A_t
$\check{R}(\mathbf{x},B) = \sum_{n=1}^\infty((\check{R}_1^n(\mathbf{x},B) - \widetilde{m}_F(B)), \quad \widetilde{m}_F(B) = m(B\cap F)/m(F)$

Let us define the recurrent potential by

$$R\mu(\mathbf{x}) = H_F \check{R}_F R^g \mu + R^g \mu(\mathbf{x}) - \frac{1}{m(F)} \langle \mu, 1 \rangle, \quad \mu \in \mathcal{M}_0$$

Then it satisfies the modified Poisson equation

$$\mathcal{E}(R\mu, u) = \langle \mu, \widetilde{u} - < \widetilde{m}_F, u \rangle \rangle, \qquad \forall u \in \mathcal{F}_e.$$

Further $R\mu(\mathbf{x})$ is expressed as

$$R\mu(\mathbf{x}) = \int r(\mathbf{x}, \mathbf{y}) \mu(d\mathbf{y}), \quad r(\mathbf{x}, \mathbf{y}) = R^g \check{R}_F r^g(\mathbf{x}, \mathbf{y}) + r^g(\mathbf{x}, \mathbf{y}) - \frac{1}{m(F)}.$$

It is a bounded linear operator on $L^1(m_F)$ and $L^\infty(m_F)$.

Equilibrium measure

- μ^B is the *equilibrium measure* of a Borel set B
  if μ^B is a probability measure on $\overline{B}$ such that $R\mu^B = c(B)$ on B
- $c(B)$ is called the *Robin's constant*.

They are characterized by

$$c(B) = \min\{\mathcal{E}(R\mu, R\mu) : \mu \text{ is a probability measure on } \overline{B}\}$$

μ^B the unique measure attaining the minimum

Explicit expressions of μ^B and $c(B)$:

$$\mu^B(A) = \mathbb{P}_{\widetilde{m}_F}(X_{\sigma_B} \in A), \quad c(B) = \frac{1}{m(F)}\mathbb{E}_{\widetilde{m}_F}\left(\int_0^{\sigma_B} 1_F(X_t)dt\right).$$

In particular, put $\mu^{\mathbf{x},r} \equiv \mu^{B(\mathbf{x},r)}$, $f(\mathbf{x},r) = c(B(\mathbf{x},r))$.

Then $f(\mathbf{x},r)$: strictly increases to ∞ as $r \downarrow 0$

Brownian motion case

$\mathbb{M}$: 2-dimensional Brownian motion, then

$$\mathcal{E}(u,v) = \frac{1}{2}\int \nabla u \cdot \nabla v d\mathbf{x}, \quad u,v \in \mathcal{F} = H_0^1(E),$$

$\nu^{\mathbf{x},r}$: uniform probability measure on $\partial B(\mathbf{x},r)$
The logarithmic potential of $\nu^{\mathbf{x},r}$ is

$$U\nu^{\mathbf{x},r}(\mathbf{y}) = \frac{1}{\pi}\log\frac{1}{|\mathbf{x}-\mathbf{y}| \vee r}.$$

Hence $\nu^{\mathbf{x},r}$: equilibrium measure,
$f_U(\mathbf{x},r) = \frac{1}{\pi}\log\frac{1}{r}$: Robin constant of $B(\mathbf{x},r)$ relative to U.

By modified Poisson equation, $R\nu^{\mathbf{x},r} - U\nu^{\mathbf{x},r}$=constant on $B(S)$.
Hence $\nu^{\mathbf{x},r}$ becomes the equilibrium measure relative to R,
$f(\mathbf{x},r) = f_U(\mathbf{x},r) - 2\langle \widetilde{m}_F, U\nu^{\mathbf{x},r}\rangle + \langle \widetilde{m}_F, U\widetilde{m}_F\rangle$: Robin constant

Gaussian field indexed by $\mathcal{F}_e$

$\{X_u : u \in \mathcal{F}_e\}$: centered Gaussian field with covariance
$$\mathbb{E}(X_u X_v) = \mathcal{E}(u, v).$$
We shall show the existence of Liouville random measure by using the method given by [N. Berestycki, Electro. Commun. Probab. 2017]

$$Y^{\mathbf{x},\varepsilon} = X_{R\mu^{\mathbf{x},\varepsilon}}, \quad \widetilde{Y}^{\mathbf{x},\varepsilon,\gamma} = \gamma Y^{\mathbf{x},r} - \frac{\gamma^2}{2} f(\mathbf{x}, \varepsilon).$$

For $\sigma(d\mathbf{x}) = \sigma(\mathbf{x})m(d\mathbf{x})$ with $0 < \sigma(\mathbf{x}) \le \|\sigma\|_\infty < \infty$ and Borel set A,

$$\mathrm{I}_\varepsilon(\omega) = \int_A e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}} \sigma(d\mathbf{x})$$

For $0 < \alpha, 0 < \varepsilon < \varepsilon_0 < 1/2$ and $\bar{r}$ satisfying $f(\mathbf{x}, \bar{r}) = [f(\mathbf{x}, r)]$,

$$G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon} = \{Y^{\mathbf{x},\bar{r}} \le \alpha f(\mathbf{x}, \bar{r}), \forall r \in (\varepsilon, \varepsilon_0)\}$$

$$\mathrm{J}_\varepsilon(\omega) = \int_A e^{\widetilde{Y}^{\mathbf{x},\varepsilon,\gamma}} 1_{G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}} \sigma(d\mathbf{x})$$

To prove the existence of the Liouville random measure

$$\overline{\mu}(A,\omega)=\lim_{\varepsilon\to 0}\mathrm{I}_\varepsilon(\omega)\qquad \text{weakly in probability},$$

prove the uniform integrability of $\mathrm{I}_\varepsilon(\omega)$ and $L^1(\mathbb{P})$-convergence.

For this, prove the $L^2(\mathbb{P})$ convergence of $\mathrm{J}_\varepsilon(\omega)$. To show $\mathbb{E}[(\mathrm{J}_\varepsilon-\mathrm{J}_\delta)^2]=\mathbb{E}[\mathrm{J}_\varepsilon^2-2\mathrm{J}_\varepsilon\mathrm{J}_\delta+\mathrm{J}_\delta^2]\to 0$ $(\varepsilon,\delta\to 0)$, by the Cameron-Martin formula,

$$\begin{aligned}\mathbb{E}[\mathrm{J}_\varepsilon\mathrm{J}_\delta] &= \int_A\int_A \mathbb{E}\left[\exp\left(\gamma(Y^{\mathbf{x},\varepsilon}+Y^{\mathbf{y},\delta}-\frac{\gamma^2}{2}(f(\mathbf{x},\varepsilon)+f(\mathbf{y},\delta)\right)\right.\\ &\qquad\left. 1_{G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}}1_{G^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}}\right]\sigma(d\mathbf{x})\sigma(d\mathbf{y})\\ &= \int_A\int_A e^{\gamma^2\mathrm{Cov}(Y^{\mathbf{x},\varepsilon},Y^{\mathbf{y},\delta})}\mathbb{P}(\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}\cap\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{y},\delta})\sigma(d\mathbf{x})\sigma(d\mathbf{y}).\end{aligned}$$

$\widetilde{G}^{\alpha,\varepsilon_0}_{\mathbf{x},\varepsilon}=\{Y^{\mathbf{x},\overline{r}}\le\alpha f(\mathbf{x},\overline{r})-\gamma\mathrm{Cov}(Y^{\mathbf{x},\varepsilon}+Y^{\mathbf{y},\delta},Y^{\mathbf{x},\overline{r}}),\forall r\in(\varepsilon,\varepsilon_0)\}.$

Used properties

To estimate it, we use the relation

$$\mathrm{Cov}(Y^{\mathbf{x},\varepsilon}, Y^{\mathbf{y},\delta}) \leq \kappa f(\mathbf{y}, (|\mathbf{x}-\mathbf{y}| - (\varepsilon+\delta)) \vee \delta) + C \tag{1}$$

for some constants $\kappa \geq 1$ and $C \geq 0$.

In the case of Brownian motion,
$f_B(\mathbf{x}, r) = U\nu^{\mathbf{x},r} + \ell(S) = \frac{1}{\pi}\log\frac{1}{r} + \ell(S)$:
Robin constant w.r.t. Brownian motion
$< \mu^{\mathbf{x},\varepsilon}, U\mu^{\mathbf{y},\delta}\rangle = \int \frac{1}{\pi}\log\frac{1}{|\mathbf{y}-\mathbf{z}|\vee\delta}\mu^{\mathbf{x},\varepsilon} + \text{bounded function.}$
Hence (1) holds with $\kappa = 1$.

The Robin constant $f(\mathbf{x}, r)$ of uniformly elliptic form is estimated by $f_B(\mathbf{x}, r)$ as

$$\frac{1}{2\Lambda}\left(\frac{1}{\pi}\log\frac{1}{r} + \ell(S)\right) \leq f(\mathbf{x}, r) \leq \frac{1}{2\lambda}\left(\frac{1}{\pi}\log\frac{1}{r} + \ell(S)\right)$$

Liouville random measure: Recurrent case

$$\mu_\varepsilon(A) = \int_A \exp\left(\gamma Y^{\mathbf{x},\varepsilon} - \frac{\gamma^2}{2} f(\mathbf{x},\varepsilon)\right)\sigma(d\mathbf{x})$$

$\rho(\mu,\nu)$: a metric inducing the weak convergence of measures on $B(S-1)$.

Theorem

If $$0 < \gamma < 2\sqrt{\frac{2\pi\lambda\Lambda}{(2\kappa\Lambda - \lambda)}}$$

then there exists a finite measure $\overline{\mu}(\cdot,\omega)$ such that, for any $\delta > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}(\rho(\mu_\varepsilon,\overline{\mu}) > \delta) = 0.$$

If $(a_{ij}) \subset \mathcal{C}^2(E)$, then we can take as $\kappa = \Lambda/\lambda$. Then the condition of the theorem holds if $\gamma \in (0, 2\sqrt{2\pi\lambda^2\Lambda/(2\Lambda^2 - \lambda^2}))$. In particular, for the Brownian motion, this condition becomes $\lambda \in (0, 2\sqrt{\pi})$.

For $D = B(S)$, let $\mathbb{M}^D$ be the part of $\mathbb{M}$ on D
$\mathbb{P}^D_{\mathbf{x}}(\sigma_B < \infty) = R^D\mu_{B(\mathbf{x},r)}$: $\mu_{B(\mathbf{x},r)}$: equilibrium measure,
$\mu^{D,\mathbf{x},r}(A) = \mu_{B(\mathbf{x},r)}(A)/\mathrm{Cap}(B(\mathbf{x},r))$: normalized equilibrium measure.

$$R^D\mu^{D,\mathbf{x},r}(\mathbf{x}) = f^D(\mathbf{x},r) = \frac{1}{\mathrm{Cap}(B(\mathbf{x},r))} \quad : \ \text{Robin constant q.e. on } B(\mathbf{x},r)$$

Relation between $R\mu$ and $R^D\mu$: If $\mathrm{supp}[\mu] \subset D$

$$R\mu - H_{E\setminus D}R\mu = R^D\mu.$$

In particular,

$$f(\mathbf{x},r) - \langle \mu^{\mathbf{x},r}_D, H_{E\setminus D}R\mu^{\mathbf{x},r}\rangle = f_D(\mathbf{x},r)$$

Further, if $u \in \mathcal{F}^D$ is $\mathbb{M}^D$-harmonic on $B(\mathbf{x},r) \subset D$, then it is $\mathbb{M}$-harmonic
Hence, κ can be taken commonly with the case of E.

Liouville random measure: Transient case

As the recurrent case, put $Y^{D,\mathbf{x},\varepsilon} = X_{R^D \mu^{D,\mathbf{x},\varepsilon}}$ and

$$\mu_\varepsilon^D(A,\omega) = \int_A \exp\left(\gamma Y^{D,\mathbf{x},\varepsilon} - \frac{\gamma^2}{2} f^D(\mathbf{x},\varepsilon)\right) \sigma(d\mathbf{x}).$$

Then, similarly to the recurrent case, we have the following result.

Theorem

For γ satisfying the condition of Theorem 1, there exists a finite measure $\overline{\mu}^D(\cdot,\omega)$ such that, for any $\delta > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}^D(\rho(\mu_\varepsilon^D, \overline{\mu}) > \delta) = 0.$$

Thank You !!