A natural extension of Markov processes and applications to singular SDEs

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1. Singular SDEs on Hilbert spaces

Let us recall the situation of [Da Prato/R.: PTRF 2002] and [Da Prato/R./Wang: JFA 2009].

$(H, \langle \ , \ \rangle)$ separable real Hilbert space with norm $|\cdot|$.

Consider the following SDE in H:

$$\begin{aligned} \mathrm{d}X(t) &= \big(AX(t) + F_0(X(t))\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \\ X(0) &= x \in H, \end{aligned} \tag{SDE}$$

where $W(t),\ t \geq 0$, is a cylindrical $(\mathcal{F}_t)$-Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $(\mathcal{F}_t)$.

Hypothesis 1

(i) $A\colon D(A) \subset H \longrightarrow H$ *is a self-adjoint linear operator which generates a* C_0*-semigroup* $T_t = e^{tA}$ *on* H*, and there exists* $\omega \in \mathbb{R}$ *such that*

$$\langle Ax, x\rangle \leq \omega |x|^2 \quad \text{for all } x \in D(A).$$

(ii) σ *is symmetric and positive definite such that* $\sigma^{-1} \in L(H)$ *and for some* $\alpha > 0$

$$\int_0^\infty (1 + t^{-\alpha}) |T_t|_{HS}^2 \mathrm{d}t < \infty,$$

where $|\cdot|_{HS}$ *denotes the Hilbert–Schmidt norm.*

(iii) F_0 *is a (possibly) nonlinear mapping given by*

$$F_0(x) := \operatorname*{arg\,min}_{y \in F(x)} |y|, \quad x \in D(F),$$

where $F\colon D(F) \subset H \longrightarrow 2^H$ *is an m-dissipative mapping, i.e.*

$$\langle u - v, x - y\rangle \leq 0 \quad \forall\, x, y \in D(F),\ u \in F(x),\ v \in F(y)$$

and $\text{Range}\,(\mathrm{I} - F) := \bigcup_{x \in D(F)} (x - F(x)) = H.$

The **Kolmogorov operator** associated to (SDE) is

$$L_0\varphi(x) = \frac{1}{2}\,\mathrm{Tr}[\sigma^2 D^2\varphi(x)] + \langle x, AD\varphi(x)\rangle + \langle F_0(x), D\varphi(x)\rangle, \quad x \in D(F), \varphi \in \mathcal{E}_A(H),$$

where $\mathcal{E}_A(H)$ is the linear space generated by the (real parts of) functions of type $\varphi(x) = \exp\{i\langle x, h\rangle\}$, $x \in H$, with $h \in D(A)$.

Hypothesis 2

There exists a Borel probability measure ν on H such that

(i) $\displaystyle\int_{D(F)} (1 + |x|^4)(1 + |F_0(x)|^2)\nu(\mathrm{d}x) < \infty.$

(ii) $\displaystyle\int_H L_0\varphi \mathrm{d}\nu = 0$ *for all* $\varphi \in \mathcal{E}_A(H)$.

(iii) $\nu(D(F)) = 1$.

Example

(i) *$H = L^2(0,1)$, $A =$ Dirichlet Laplacian on $(0,1)$, $F_0(x) := -p(x)$, $x \in D(F_0) := L^{2m}(0,1)$, where p is an increasing polynomial of order m.*

(ii) *Further examples in [Da Prato/R.: PTRF 2002], [Da Prato/R./Wang: JFA 2009].*

Notation: $H_0 := \text{supp}(\nu)$, $Lip_b(H_0) :=$ all real-valued bdd. Lipschitz-functions on H_0, $\mathbb{B}_b(H_0)$:=all real-valued bdd. Borel-measurable functions on H_0.

1.1 Known results

Theorem 0

The following assertions hold.

(i) *(cf. [Da Prato/R.: PTRF 2002])* $(L_0, \mathcal{E}_A(H))$ *is closable on* $L^2(\nu) := L^2(H, \nu)$*, its closure denoted by* $(L, D(L))$ *is m-dissipative and:*

- (i.1) *there exists a Lipschitz strong Feller Markovian semigroup of kernels on* H_0 *denoted by* $(P_t)_{t\geq 0}$ *such that* $\lim_{t\to 0} P_t f = f$ *pointwise on* H_0 *for all* $f \in Lip_b(H_0)$*; by (Lipschitz) strong Feller we mean* $P_t(\mathcal{B}_b(H_0)) \subset C_b(H_0)$ *(resp.* $Lip_b(H_0)$*).*
- (i.2) ν *is invariant for* $(P_t)_{t\geq 0}$ *and the extension of* $(P_t)_{t\geq 0}$ *to* $L^2(\nu)$ *is the strongly continuous semigroup generated by* L*.*

(ii) *(cf. [Da Prato/R./Wang: JFA 2009])* ν *satisfying Hypothesis 2 is unique,* $P_t(L^q(H, \nu)) \subset C(H_0)$*, and the following Harnack inequality holds*

$$(P_t f(x))^q \leq P_t f^q(y) e^{|\sigma^{-1}|^2 \frac{p\omega |x-y|^2}{(q-1)(1-e^{-2\omega t})}}$$

for all $f \geq 0$, $t > 0$, $q \in (1, \infty)$, $x, y \in H_0$*. In particular,* $P_t(\mathrm{d}x) \ll \nu$, $t > 0$*.*

Theorem 0 (continued)

(iii) *There exists* $M \in \mathcal{B}(H_0)$ *such that* **for each** $\boldsymbol{x} \in \boldsymbol{M}$ *there exists a pathwise unique continuous strong solution* $X(t,x)$, $t \geq 0$, *(in the mild sense) for* (SDE) *starting* **from** $\boldsymbol{x} \in \boldsymbol{M}$ *such that*

$$\mathbb{P}(X(t,x) \in M \ \ \forall\, \boldsymbol{t} \geq \mathbf{0}) = 1$$

and

$$\mathbb{E}[f(X(t,x))] = P_t f(x), \ x \in M, \ f \in \mathcal{B}_b(H_0).$$

1.2 New results

Theorem I ([Beznea/Cimpean/R.: arXiv 2019])

The following assertions hold:

(i) *Let M be as in Theorem 0. For every $x \in H_0 \setminus M$ there exists a generalized solution $X(t,x)$, $t \geq 0$ starting from x (in the sense of [Da Prato/Zabczyk 2014]).*

(ii) *There exists a conservative right (strong) Markov process $X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X(t))_{t\geq 0}, (\theta(t))_{t\geq 0}, (\mathbb{P}^x)_{x\in H_0})$ on H_0 (see definition below) with a.s. $|\cdot|$-continuous paths and transition semigroup $(P_t)_{t\geq 0}$. In particular*

$$\mathbb{P}^x \circ X(\cdot)^{-1} = \mathbb{P} \circ X(\cdot, x)^{-1} \quad \text{for all } x \in H_0.$$

In addition, the following assertions hold:

(ii.1) **For all $x \in H_0$** *we have $\mathbb{P}^x(X(t) \in M$ for all $t > 0) = 1$ ("$H_0 \setminus M$ is polar"), where M is the set from (i).*

(ii.2) **For every $x \in H_0$**, *$\mathbb{P}_x$ solves the martingale problem for L with test function space*

$$D_0 := \{\varphi \in D(L) \cap C_b(H) |\ L\varphi \in L^\infty(H, \nu)\}$$

and initial condition x, i.e. $\mathbb{P}_x$-a.s. $X(0) = x$ and

$$\varphi(X(t)) - \varphi(X(0)) - \int_0^t L\varphi(X(s))\mathrm{d}s, \quad t \geq 0,$$

is a continuous $(\mathcal{F}_t)$-martingale for all $\varphi \in D_0$.

Theorem I (continued)

(ii.3) **If $\boldsymbol{x} \in \boldsymbol{H_0} \setminus \boldsymbol{M}$** *and* $\varepsilon > 0$, *then under* $\mathbb{P}^x$ *it holds that* $(X(t+\varepsilon))_{t\geq 0}$ *is a probabilistically weak solution to* (SDE) *(in the mild sense) starting from* $X(\varepsilon)$.

(iii) **If $\boldsymbol{x} \in \boldsymbol{H_0} \setminus \boldsymbol{M}$** *and* $\varepsilon > 0$ *is fixed, then* (SDE) *has a pathwise unique continuous strong solution with initial distribution* $\mathbb{P}^x \circ X(\varepsilon)^{-1}$.

Remark

Obviously, since X *is a Markov process with transition semigroup* $(P_t)_{t\geq 0}$, *the laws* $\mathbb{P}^x \circ X(\cdot)^{-1}$, $x \in H_0$, *are uniquely determined by these two properties. So, indeed we have* $\mathbb{P}^x \circ X(\cdot)^{-1} = \mathbb{P} \circ X(\cdot, x)^{-1}$, *if* $x \in H_0$.

Remark

(i) *In fact, result (ii.2) was also claimed in [Da Prato/R.: PTRF 2002]. However, there was a mistake in the proof that*

$$\mathbb{P}^x\big(C([0,\infty);H_0)\big) = 1 \quad \text{for all } x \in H_0$$

and only

$$\mathbb{P}^x\big(C((0,\infty);H_0)\big) = 1 \quad \text{for all } x \in H_0$$

was correctly proved (see [Da Prato/R.: PTRF 2009]). So, by Theorem I above, all claims in [Da Prato/R.: PTRF 2002] are finally proved.

(ii) *The proof of Theorem I is an application of a general extension result for Markov processes which will be presented in the next section.*

2. Excursion on right processes

$(E, \mathcal{B})$ = Lusin measurable space (i.e. measurable isomorphic to a Borel subset of a compact metric space).

Recall: Topology τ on E is called Lusin if (E, τ) homeomorphic to a Borel subset of a polish space.

Let $X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X(t))_{t \geq 0}, (\theta(t))_{t \geq 0}, (\mathbb{P}^x)_{x \in E})$ be a normal Markov process with state space E and shift operators $\theta \colon \Omega \longrightarrow \Omega$, $t \geq 0$. Its corresponding resolvent $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ is defined by

$$U_\alpha f(x) = \mathbb{E}^x \Big[\int_0^\infty e^{-\alpha t} f(X(t)) \, \mathrm{d}t \Big], \quad x \in E.$$

For $\beta > 0$ set $\mathcal{U}_\beta := (U_{\alpha+\beta})_{\alpha > 0}$.

Definition 1

A $\mathcal{B}$-measurable function $v: E \longrightarrow \bar{\mathbb{R}}_+$ is called **excessive** *(w.r.t. $\mathcal{U}$) if $\alpha U_\alpha v \leq v$ for all $\alpha > 0$ and $\sup_\alpha \alpha U_\alpha v = v$ pointwise; by $\mathcal{E}(\mathcal{U})$ we denote the convex cone of all excessive functions w.r.t. $\mathcal{U}$.*

Definition 2

(i) *The* **fine topology** *on E (associated with $\mathcal{U}$) is the coarsest topology on E such that every $\mathcal{U}_\beta$-excessive function is continuous for some (hence all) $\beta > 0$.*

(ii) *A topology τ on E is called* **natural** *if it is a Lusin topology which is coarser than the fine topology, and whose Borel σ-algebra is $\mathcal{B}$.*

Remark

The necessity of considering natural topologies comes from the fact that, in general, the fine topology is neither metrizable, nor countably generated.

To each probability measure μ on $(E, \mathcal{B})$ we associate the probability $\mathbb{P}^{\mu}(A) := \int \mathbb{P}^{x}(A)\mu(\mathrm{d}x)$ for all $A \in \mathcal{F}$, and we consider the following enlarged filtration

$$\widetilde{\mathcal{F}}_t := \bigcap_{\mu} \mathcal{F}_t^{\mu}, \; \widetilde{\mathcal{F}} := \bigcap_{\mu} \mathcal{F}^{\mu},$$

where $\mathcal{F}^{\mu}$ is the completion of $\mathcal{F}$ under $\mathbb{P}^{\mu}$, and $\mathcal{F}_t^{\mu}$ is the completion of $\mathcal{F}_t$ in $\mathcal{F}^{\mu}$ w.r.t. $\mathbb{P}^{\mu}$.

Definition 3

The Markov process X is called a right (Markov) process if the following additional hypotheses are satisfied:

(i) *The filtration $(\mathcal{F}_t)_{t \geq 0}$ is right continuous and $\mathcal{F}_t = \widetilde{\mathcal{F}}_t$, $t \geq 0$.*

(ii) *For one (hence all) $\beta > 0$ and for each $f \in \mathcal{E}(\mathcal{U}_{\beta})$ the process $f(X)$ has right continuous paths $\mathbb{P}^{x}$-a.s. for all $x \in E$.*

(iii) *There exists a natural topology on E with respect to which the paths of X are $\mathbb{P}^{x}$-a.s. right continuous for all $x \in E$.*

Fact: If X is a right process, then it has a.s. right continuous paths w.r.t. any natural topology on E.

3. A natural extension of a Markov process

$(E, \mathcal{B})$ = Lusin measurable space.

Let $M \in \mathcal{B}$ and $X = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X(t))_{t\geq 0}, (\theta(t))_{t\geq 0}, (\mathbb{P}^x)_{x\in E})$ a right Markov process with state space M with resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$.

Definition 4

We say that a Markov process $\overline{X} = (\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathcal{F}}_t, \overline{X}(t), \overline{\theta}(t), \overline{\mathbb{P}}^x)$, *with state space* E, *is a* **natural extension** *of* X *if the following conditions are fulfilled.*

(i) $\overline{X}$ *is a right process.*

(ii) *The processes* $((X(t))_{t\geq 0}, \mathbb{P}^x)$ *and* $((\overline{X}(t))_{t\geq 0}, \overline{\mathbb{P}^x})$ *are equal in distribution for all* $x \in M$*;*

(iii) *For every* $x \in E$ *one has* $\overline{\mathbb{P}^x}$*-a.s.* $\overline{X}(t) \in M$ *for all* $t > 0$*, i.e.* $E \setminus M$ *is polar w.r.t.* $\overline{X}$*.*

Definition 5

A sub-Markovian resolvent of kernels $\overline{\mathcal{U}} := (\overline{U}_\alpha)_{\alpha>0}$ *on* E *(i.e. each* $\overline{U}_\alpha$ *is a kernel on* $(E, \mathcal{B})$ *such that for all* $\alpha, \beta > 0$*:* $\overline{U}_\alpha - \overline{U}_\beta = (\alpha - \beta)\overline{U}_\alpha\overline{U}_\beta$, $\alpha\overline{U}_\alpha 1 \leq 1$*) is called an extension of* $\mathcal{U}$ *if:*

(i) $\overline{U}_\alpha(1_{E\setminus M}) = 0$.

(ii) $(\overline{U}_\alpha f)|_M = U_\alpha(f|_M)$ *(on* M*) for all* $\alpha > 0$ *and* $f \in \mathcal{B}_b$.

Easy fact: If $\overline{X}$ is a natural extension of X, then its resolvent denoted by $\overline{\mathcal{U}}$ is an extension of $\mathcal{U}$.

Question: When does the converse hold?

Consider the following condition:

(H) There exists a min-stable convex cone $\mathcal{C} \subset \mathcal{B}_b^+$ (= all nonnegative bounded $\mathcal{B}$-measurable functions) such that

(i) $1 \in \mathcal{C}$ and $\sigma(\mathcal{C}) = \mathcal{B}$.

(ii) For some (hence all) $\alpha > 0$ we have $\overline{U}_\alpha f \in \mathcal{C}$ for all $f \in \mathcal{C}$.

(iii) $\lim_{\alpha \to \infty} \alpha \overline{U}_\alpha f = f$ point-wise on E for all $f \in \mathcal{C}$.

Then:

Theorem II ([Beznea/Cimpean/R.: arXiv 2019])

(i) *Let $\overline{\mathcal{U}}$ be an extension of $\mathcal{U}$. Then there exists a natural extension $\overline{X}$ of X, with resolvent $\overline{\mathcal{U}}$, if and only if (H) is satisfied.*

(ii) *Any extension $\overline{\mathcal{U}}$ of $\mathcal{U}$ which satisfies (H), is uniquely determined. In particular, any natural extension of X is unique in distribution.*

Corollary

Theorem I (ii) holds.

Proof.

(H) holds with $C = Lip_b(H_0)$. □

4. Singular SDEs on Hilbert spaces perturbed by a bounded drift B

Let $B\colon H \longrightarrow H$ be Borel measurable and bounded. Consider following SDE in H:

$$\begin{aligned} \mathrm{d}X(t) &= (AX(t)) + F_0(X(t)))\,\mathrm{d}t \boldsymbol{+B(X(t))\,\mathrm{d}t} + \sigma \mathrm{d}W(t) \\ X(0) &= x \in H \end{aligned} \qquad (\mathrm{SDE}_{\mathbf{B}})$$

The corresponding Kolmogorov operator is

$$\begin{aligned} L^B\varphi(x) = \frac{1}{2}\,\mathrm{Tr}\,[\sigma^2 D^2\varphi(x)] + \langle x, AD\varphi(x)\rangle + \langle F_0(x), D\varphi(x)\rangle \\ \boldsymbol{+\langle B(x), D\varphi(x)\rangle}, \qquad x \in D(F),\ \varphi \in \mathcal{E}_A(H). \end{aligned}$$

Let us fix a cylindrical $(\mathcal{F}_t)$-Wiener process $\widetilde{W}$ on a stochastic basis $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t), \widetilde{\mathbb{P}})$ with normal filtration $(\widetilde{\mathcal{F}}_t)$, and take $(X(t,x))_{t\geq 0}$ to be the **generalized solution** given by Theorem I (i). For each $t > 0$, we define the Markov kernels

$$Q_t f(x) := \mathbb{E}^{\widetilde{\mathbb{P}}}\{f(X(t,x))\rho_t^x\}$$

for all $f \in \mathcal{B}_b(H_0)$ and $x \in H_0$, where

$$\rho_t^x := e^{\int_0^t \langle B(X(s,x)), d\widetilde{W}(s)\rangle - \frac{1}{2}\int_0^t |B|^2(X(s,x))ds}$$

are continuous $(\widetilde{\mathcal{F}}_t)$-martingales by Novikov's condition.
Let $\mathcal{V} := (V_\alpha)_{\alpha>0}$ denote the resolvent of kernels associated to $(Q_t)_{t\geq 0}$, i.e. for $\alpha > 0$ and $f \in \mathcal{B}_b(H_0)$

$$V_\alpha f(x) := \int_0^\infty e^{-\alpha t} Q_t f(x) dt, \quad x \in H_0.$$

Theorem III ([Beznea/Cimpean/R.: arXiv 2019])

There exists a conservative right Markov process $Y = (\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, (Y(t))_{t\geq 0}, (\theta(t))_{t\geq 0}, (\mathbb{Q}^x)_{x\in H_0})$ *on* H_0 *with a.s.* $|\cdot|$*-continuous paths, transition function* $(Q_t)_{t\geq 0}$*, and Lipschitz strong Feller resolvent* $\mathcal{V}$*. In addition, the following assertions hold:*

(i) $(Q_t)_{t\geq 0}$ *extends to a strongly continuous semigroup on* $L^2(\nu)$*, whose infinitesimal generator* $(L^B, D(L^B))$ *is the closure of* $(L_0^B, \mathcal{E}_A(H))$ *on* $L^2(\nu)$*; in particular,* $D(L^B) = D(L)$*.*

(ii) **For every** $\boldsymbol{x \in H_0}$, $\mathbb{Q}_x$ *solves the martingale problem for* L^B *with the same test function space as in Theorem I and initial condition* x*, i.e.* $Y(0) = x$ $\mathbb{Q}_x$*-a.s. and under* $\mathbb{Q}_x$

$$\varphi(Y(t)) - \varphi(Y(0)) - \int_0^t L^B\varphi(Y(s))ds, \quad t \geq 0,$$

is a continuous $(\mathcal{F}_t)$*-martingale for all* $\varphi \in D_0$*.*

(iii) **If** $\boldsymbol{x \in M}$, *then under* $\mathbb{Q}^x$*, the Markov process* Y *is a (unique in law) probabilistically weak solution to* $(\mathrm{SDE}_{\mathbf{B}})$ *(in the mild sense),* **which remains in** $\boldsymbol{M}$*.*

(iv) **If** $\boldsymbol{x \in H_0 \setminus M}$ *and* $\varepsilon > 0$*, then under* $\mathbb{Q}^x$ *we have that* $(Y(t+\varepsilon))_{t\geq 0}$ *is a probabilistically weak solution to* $(\mathrm{SDE}_{\mathbf{B}})$ *(in the mild sense) starting from* $Y(\varepsilon)$*.*

Remark

Feller properties of $\mathcal{V} = (V_\alpha)_{\alpha>0}$ are hard to obtain from its definition since B is not continuous. They are proved here by an analytic perturbation argument coming from the special form of the Kolmogorov operator L^B (generalizing the one in [Da Prato/ Flandoli/R./Veretennikov: AOP 2016]).