Tail estimate for random walk in random scenery

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Random walk in random scenery (RWRS)

- $(\{z(x)\}_{x\in\mathbb{Z}^d}, \mathbb{P})$: IID random variables,
- $(S_n)_{n\in\mathbb{Z}_+}$: Random walk on $\mathbb{Z}^d$.

Random walk in random scenery:

$$W_n := \sum_{k=1}^{n} z(S_k).$$

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- Introduced by Borodin and Kesten-Spitzer in 1979.
- Scaling limit (under $\mathbb{P} \otimes P_0$) yields a self-similar process.

  $d = 1$, z: α-stable, S: $\beta_{>1}$-stable $\Rightarrow$ index $= 1 - \frac{1}{\beta} + \frac{1}{\alpha\beta}$.
- Central limit theorem hold if S is transient and $\mathrm{Var}(z) < \infty$.
- $d = 2$, $S = \mathrm{SRW}$, $\mathrm{Var}(z) < \infty$, $\frac{1}{\sqrt{n \log n}} W_n \to \mathcal{N}(0, \sigma^2)$ (Bolthausen 1989).

RWRS: continuous time

In this talk,

- $(\{z(x)\}_{x\in\mathbb{Z}^d}, \mathbb{P})$: IID, ≥ 0 with $\mathbb{P}(z(x) \geq r) = r^{-\alpha+o(1)}$,
- $((S_t)_{t\geq 0}, (P_x)_{x\in\mathbb{Z}^d})$: continuous time simple random walk.

Continuous time version of RWRS:

$$A_t := \int_0^t z(S_u)\mathrm{d}u.$$

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Naturally appears in random media models:

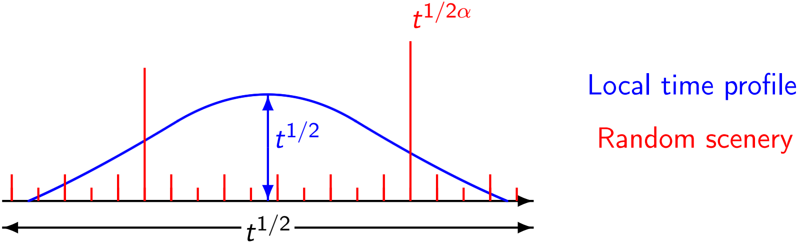
- $E_x\left[f(S_t)e^{A_t}\right]$ is a solution of $\partial_t u = \Delta u + zu$, $u(0,x) = f(x)$.
- $(S_t^1, S_t^2 + A_t^1)_{t\geq 0}$: diffusion in random shear flow.
- $(S_{A_t^{-1}})_{t\geq 0}$: random hopping time dynamics.
- $(S^1_{A_t^2}, S_t^2)_{t\geq 0}$: random walk in layered conductance.

Scaling and strategy

The typical behavior of RWRS is

$$A_t = t^{s(d,\alpha)+o(1)} \text{ with } s(d,\alpha) = \begin{cases} \frac{\alpha+1}{2\alpha} \vee 1 & \text{if } d = 1, \\ \frac{1}{\alpha} \vee 1 & \text{if } d \geq 2. \end{cases}$$

Schematic picture for $d = 1$ and $\alpha < 1$:



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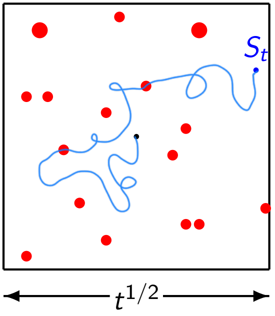
Schematic picture for $d \geq 3$ and $\alpha < 1$:



RW trajectory

- $= t^{1/\alpha}$ high points
- $= t^{d/2\alpha}$ high points

Known tail estimates for RWRS

Many *annealed* results: $\mathbb{P} \otimes P_0(A_t \geq t^\rho)$.

- Asselah, Castell, Csáki, Fleischman, Gantert, van der Hofstad, Khoshnevisan, König, Lewis, Li, Mörters, Pradeilis, Remillard, Shi, Wachtel...
- Related to law of iterated logarithm, intersection local time.

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Few *quenched* results: $P_0(A_t \geq t^\rho)$ for typical z.

- Brownian motion in Gaussian scenery,
  - Large deviation for $\frac{1}{t\sqrt{\log t}} A_t$: Asselah-Castell (2003),
  - Moderate deviations: Castell (2004),
- Brownian motion in bounded scenery,
  - Large deviation for $\frac{1}{t} A_t$: Asselah-Castell (2003).

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In particular, no results in Borodin and Kesten-Spitzer setting.

Upper tail — stretched exponential decay

Theorem (Deuschel–F. 2019)

Assume $\mathbb{P}(z(x) \geq r) = r^{-\alpha+o(1)}$ *and let* $\rho > 0$. *Then*

$$P_0\left(A_t \geq t^{\rho}\right) = \exp\left\{-t^{p(\alpha,\rho)+o(1)}\right\}, \quad \mathbb{P}\text{-almost surely}$$

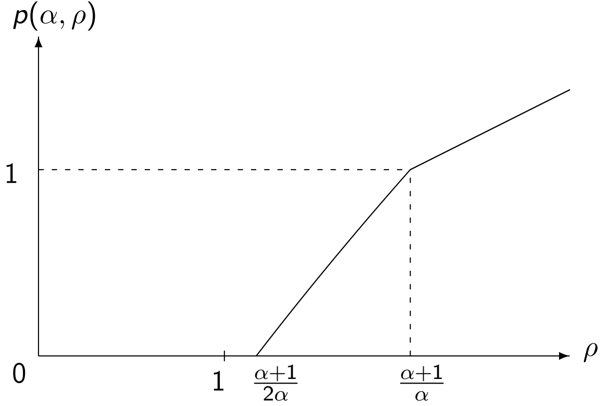
as $t \to \infty$, *where for* $d = 1$,

$$p(\alpha,\rho) = \begin{cases} \frac{2\alpha\rho}{\alpha+1} - 1, & \rho \in \left(\frac{\alpha+1}{2\alpha} \vee 1, \frac{\alpha+1}{\alpha}\right], \\ \alpha(\rho-1), & \rho > \frac{\alpha+1}{\alpha} \end{cases}$$

and for $d = 2$,

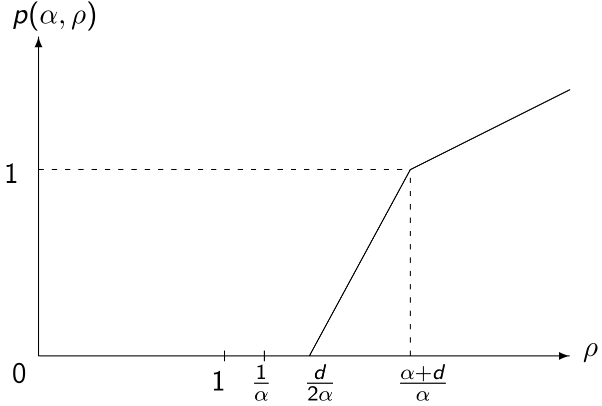
$$p(\alpha,\rho) = \begin{cases} \frac{2\alpha\rho-d}{2\alpha+d}, & \rho \in \left(\frac{d}{2\alpha} \vee 1, \frac{\alpha+d}{\alpha}\right], \\ \frac{\alpha(\rho-1)}{d}, & \rho > \frac{\alpha+d}{\alpha}. \end{cases}$$

Illustration: $d = 1, \alpha \leq 1$



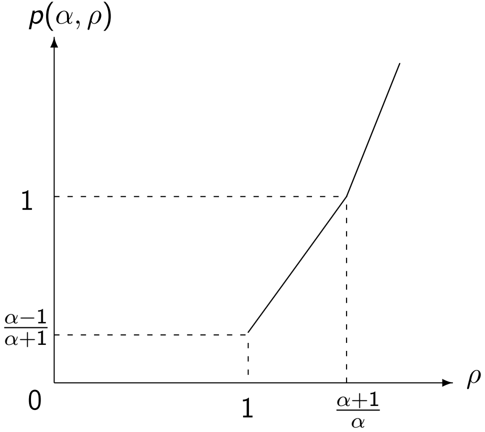
$\frac{\alpha+1}{2\alpha}$ = the typical scale for RWRS.

Illustration: $d \geq 2, \alpha \leq \frac{d}{2}$



$\frac{d}{2\alpha} > \frac{1}{\alpha}$ = the typical scale for RWRS for $d \geq 3$.

Illustration: $d = 1, \alpha > 1$ ($d \geq 2, \alpha > \frac{d}{2}$ is similar)



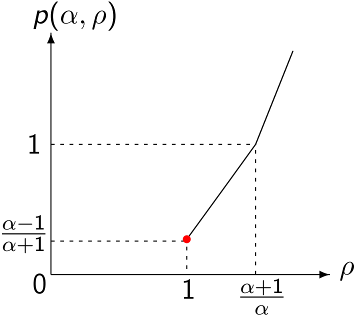
Large deviation

Theorem (Deuschel–F. 2019)

Assume $\mathbb{P}(z(x) \geq r) = r^{-\alpha+o(1)}$ *and let* $d = 1$ *and* $\alpha > 1$ *or* $d \geq 2$ *and* $\alpha > \frac{d}{2}$. *Then for any* $c > \mathbb{E}[z(x)]$, $\mathbb{P}$*-almost surely,*

$$P_0\left(A_t \geq ct\right) = \begin{cases} \exp\left\{-t^{\frac{\alpha-1}{\alpha+1}+o(1)}\right\}, & d = 1, \\ \exp\left\{-t^{\frac{2\alpha-d}{2\alpha+d}+o(1)}\right\}, & d \geq 2 \end{cases}$$

as $t \to \infty$.



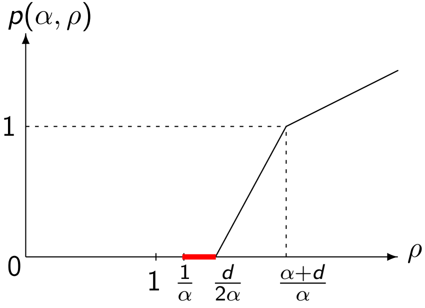
Upper tail — power law decay

Theorem (Deuschel–F. 2019+)

Assume $\mathbb{P}(z(x) \geq r) = r^{-\alpha+o(1)}$ *and let* $d \geq 3$ *and* $\alpha \leq \frac{d}{2}$*. Then for any* $\rho \in (\frac{1}{\alpha} \vee 1, \frac{d}{2\alpha})$*,* $\mathbb{P}$*-almost surely,*

$$P_0(A(t) \geq t^{\rho}) = t^{-\alpha\rho+1+o(1)}$$

as $t \to \infty$*.*



Lower tail estimate

Theorem (Deuschel–F. 2019+)

Let $d \geq 1$ and $\epsilon > 0$. Then $\mathbb{P}$-almost surely,

$$P_0\left(A(t) \leq t^{s(d,\alpha)-\epsilon}\right) \leq \exp\{-t^{c(\epsilon)}\}.$$

Moreover, if $\mathbb{E}[z(0)] < \infty$,

$$P_0\left(A(t) \leq t(\mathbb{E}[z(0)] - \epsilon)\right) \leq \exp\{-t^{c(\epsilon)}\}.$$

Remark

We only know $c(\epsilon) > 0$. More information as a function of ϵ is desirable.

Random layered conductance model

Let $((X_t)_{t\geq 0}, (P_x^\omega)_{x\in\mathbb{Z}^{1+d}})$ be a continuous time Markov chain on $\mathbb{Z}^2$ with jump rates

$$\omega(x, x \pm \mathbf{e}_i) = \begin{cases} z(x_2), & i = 1, \\ 1, & i = 2. \end{cases}$$

By using CTSRW (S^1, S^2) on $\mathbb{Z}^2$,

$$(X_t^1, X_t^2)_{t\geq 0} = (S^1_{A_t^2}, S_t^2)_{t\geq 0} \text{ with } A_t^2 = \int_0^t z(S_u^2)\mathrm{d}u.$$

Transition probability:

$$\begin{aligned} P_0^\omega\left(X_t = (x, y)\right) &= P_0\left(S^1_{A_t^2} = x, S_t^2 = y\right) \\ &= E_0\left[p_{A_t^2}(0, x) \colon S_t^2 = y\right]. \end{aligned}$$

On-diagonal estimate

Theorem

If $\alpha \leq 1$ and $\mathbb{E}[z(0)] = \infty$, then $\mathbb{P}$-almost surely,

$$P_0^\omega(X_t = 0) = t^{-\frac{3\alpha+1}{4\alpha}+o(1)}$$

as $t \to \infty$. If $\mathbb{E}[z(0)] < \infty$, then $\mathbb{P}$-almost surely,

$$P_0^\omega(X_t = 0) = \left((4\pi)^{-\frac{d+1}{2}}\mathbb{E}[z(0)]^{-\frac{1}{2}} + o(1)\right) t^{-1}$$

as $t \to \infty$. In particular, $(X_t)_{t\geq 0}$ is transient if $\alpha < 1$ and recurrent if $\mathbb{E}[z(0)] < \infty$.

Remark

There are higher dimensional analogues and off-diagonal estimates.

Thank you!