Limiting distributions for the maximal displacement of branching Brownian motions

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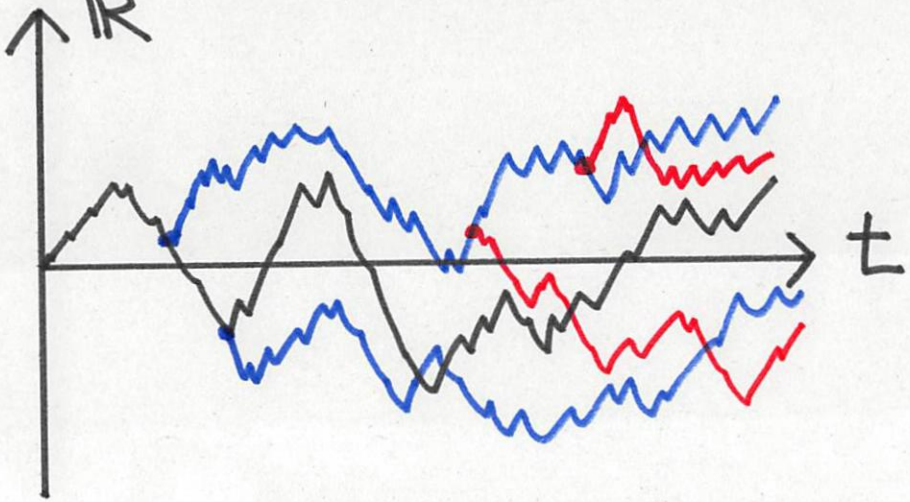
1. Introduction



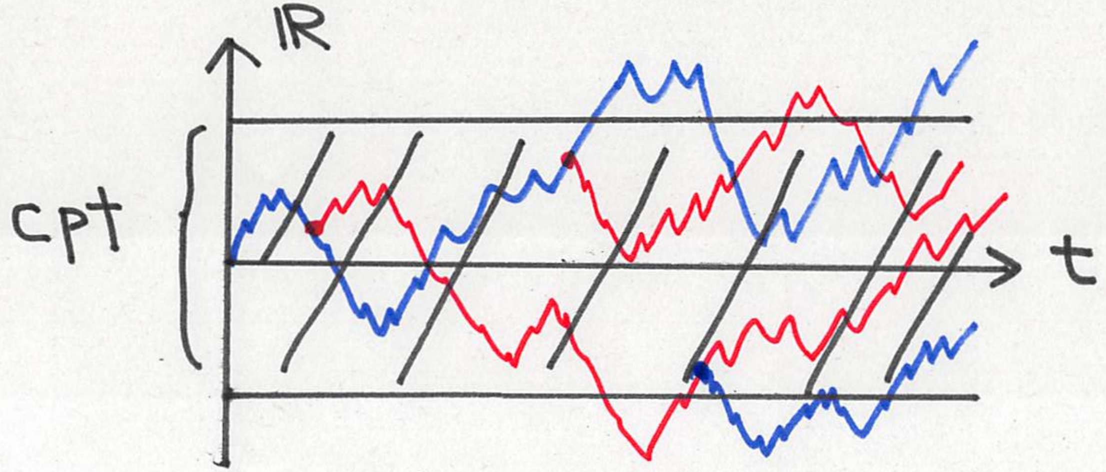
Main interest

Trajectory of the maximal norm of particles

spatial motions — population growth

interaction

· Spatially inhomogeneous model



Typical case

branching only on a cpt set

∘ $d = 1$ ($\rightsquigarrow$ BM: recurrent)

· Erickson (84): $\exists \lim_{t\to\infty} \frac{R_t}{t} =: c_*$ non-random, positive

· Lalley-Sellke (88):

$$R_t = c_* t + Y_t$$

limiting dist.

Purpose of this talk

To discuss

(1) fluctuation of R_t for $\underline{\underline{d=1,2}}$

$$R_t = c_* t + \boxed{c_{**}(d-1)\log t} + Y_t$$

dimension-dependent

limiting dist.

- $d=1,2 \rightsquigarrow$ BM: recurrent
- Bocharov-Harris (16): catalytic BBM

d=1. branching only on dot

(2) upper deviation type asymptotics

For $d \geq 1$, $\forall\ \underline{\delta > c_*}$, ← subcritical

$$\mathbb{P}_{\boxed{x}}(R_t \geq \delta t)$$

$$\sim \boxed{}$$

· exp. decay

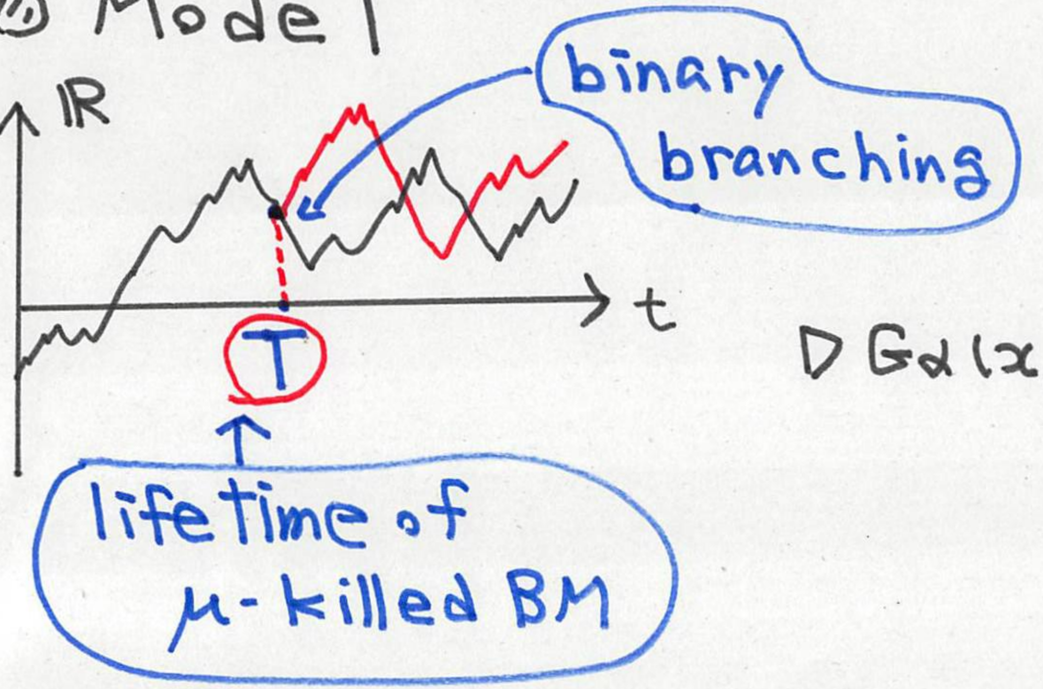
· x- dependent

· critical → poly. decay

∘ Chauvin-Rouault : spatially homogeneous model (88, 90)

2. Results

(iii) Model



▷ μ: positive Radon meas. on $\mathbb{R}^d$ with cpt supp

▷ $G_\alpha(x, y)$: α-resolvent of BM on $\mathbb{R}^d$

- Dirac meas. $(d=1)$
- surface meas. $(d \geq 2)$
- ⋮

· μ belongs to the Kato class

$$\overset{\text{def}}{\iff} \quad \lim_{\alpha \to \infty} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} G_\alpha(x, y)\, \mu(dy) = 0$$

$$-\frac{1}{2}\Delta \longrightarrow -\frac{1}{2}\Delta + \underline{\mu} \quad \text{killing}$$

binary branching

$$\longrightarrow -\frac{1}{2}\Delta + \underline{\mu} - \underline{\underline{2\mu}} = -\frac{1}{2}\Delta - \mu =: \underline{\underline{\mathcal{L}^\mu}}$$

creation

- $\lambda := \inf \left\{ \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx - \int_{\mathbb{R}^d} u^2 d\mu \;\middle|\; u \in C_0^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 dx = 1 \right\}$

  : bottom of the spectrum of $\underline{\underline{\mathcal{L}^\mu}}$

Takeda (03, 08)

$$\lambda < 0 \Rightarrow \begin{cases} \cdot\ \lambda : \text{principal eigenvalue} \\ \updownarrow \\ \cdot\ h(x) \asymp G_{-\lambda}(0, x) \asymp \dfrac{e^{-\sqrt{-2\lambda}\,|x|}}{|x|^{\frac{d-1}{2}}} \\ \qquad (|x| \gg 1) \end{cases}$$

(iii) Relation between <u>BBM</u> and <u>$\mathcal{L}^\mu$</u>

▷ Z_t := # of particles at time t

▷ $\mathbb{B}_t := (B_t^1, B_t^2, \ldots, B_t^{Z_t})$: position of particles at time t

▷ $Z_t(f) := \sum_{k=1}^{Z_t} f(B_t^k)$ (f : bdd Borel on $\mathbb{R}^d$)

$$\Rightarrow \mathbb{E}_x[Z_t(f)] = E_x[\underline{\underline{e}}^{A_t^\mu} f(B_t)] \quad (= e^{-t\mathcal{L}^\mu} f(x))$$

$\rightsquigarrow$ $\mathbb{E}_x[Z_t(h)] = e^{-\lambda t} h(x)$

$[e^{-t\mathcal{L}^\mu} h = e^{-\lambda t} h]$

▷ $M_t := e^{\lambda t} Z_t(\mathbb{R})$

$\Rightarrow$ M_t: non-negative, square integrable martingale

$\rightsquigarrow$ $^\exists \lim_{t\to\infty} M_t =: M_\infty$, $\mathbb{P}_x(M_\infty > 0) > 0$

<u>Remark</u> (S 18, 19)

- $d = 1, 2 \Rightarrow \mathbb{P}_x(M_\infty > 0) = 1$
- $d \geq 3 \Rightarrow \mathbb{P}_x(M_\infty > 0) \in (0, 1)$

$^\exists \lim_{t\to\infty} \frac{1}{t} \log Z_t = -\lambda$ on $\{M_\infty > 0\}$

"regular growth event"

▷ $R_t := \max_{1 \le k \le Z_t} |B_t^k|$

$$\cdot\ R_t = \underline{\sqrt{-\lambda/2}\, t} + \underline{\underline{\frac{d-1}{2\sqrt{-2\lambda}} \log t}} + Y_t$$

<u>Theorem 1</u>

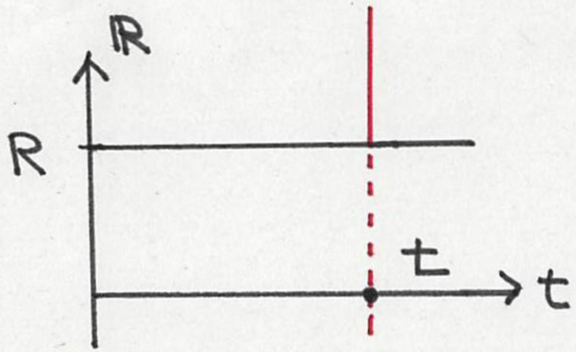
$\cdot\ d = 1, 2$ ($\rightsquigarrow$ $\lambda < 0$, $\mathbb{P}_x(M_\infty > 0) = 1$)

$\Rightarrow$ ${}^\forall \underline{\underline{x \in \mathbb{R}}}$,

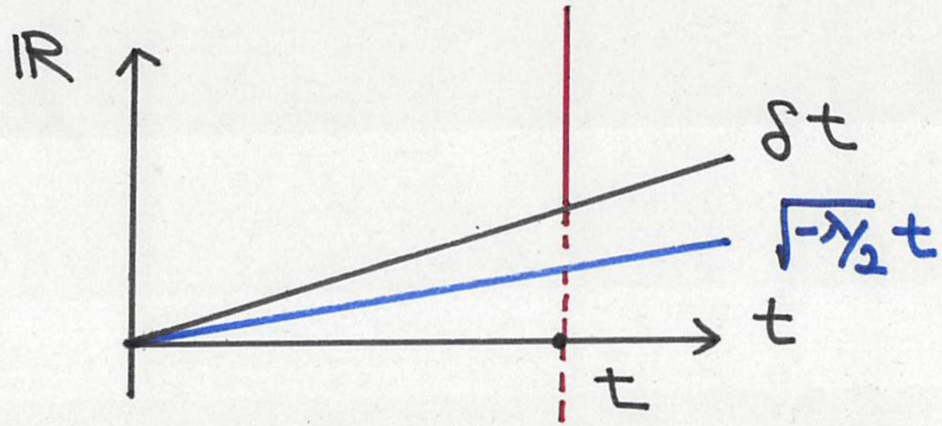
Gumbel type dist.

$$\mathbb{P}_x(Y_t \le x) \longrightarrow \mathbb{E}_x\left[e^{-M_\infty \cdot e^{-\sqrt{-2\lambda}\, x}}\right]$$

$(t \to \infty)$



▷ $Z_t^R :=$ # of particles on $\{x \in \mathbb{R}^d \mid |x| > R\}$ at time t



Theorem 2

Assume $\lambda < 0$

· $\delta \in \left(\sqrt{-\lambda/2}, \sqrt{-2\lambda}\right)$

$$\Rightarrow \mathbb{P}_x (R_t > \delta t) \sim \mathbb{E}_x [Z_t^{\delta t}]$$

$$\sim c_d \, \delta^{\frac{d-1}{2}} \underbrace{e^{(-\lambda - \sqrt{-2\lambda}\,\delta)t}\, t^{\frac{d-1}{2}}}_{\text{exp. decay}} h(x)$$

[explicit const. including h, λ]

3. Sketch of the proofs

Key Proposition

- $\delta \in (0, \sqrt{-2\lambda})$

$$\Rightarrow \boxed{\mathbb{E}_x[e^{A_t^\mu} ; |B_t| > \delta t]} \;\; // \;\; \mathbb{E}_x[Z_t^{\delta t}]$$

$\exists c > 0$

$$= e^{-\lambda t} h(x) \int_{|y| > \delta t} h(y)\,dy + \underline{O(e^{-\boxed{c}t})}$$

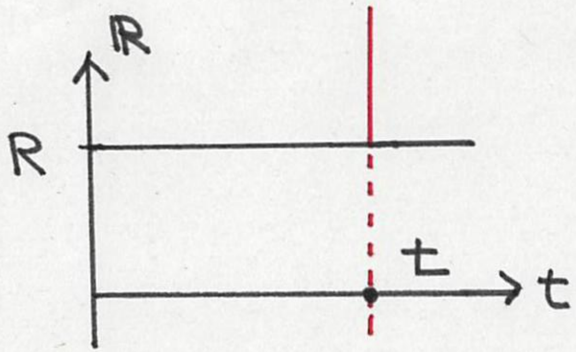
[spectral gap of $\mathcal{L}^\mu = -\frac{1}{2}\Delta - \mu$]

Takeda ('08)

- $R > 0$: const.

$$\Rightarrow \mathbb{E}_x[e^{A_t^\mu} ; |B_t| > R] \sim e^{-\lambda t} h(x) \int_{|y| > R} h(y)\,dy$$

Fukushima's ergodic theorem



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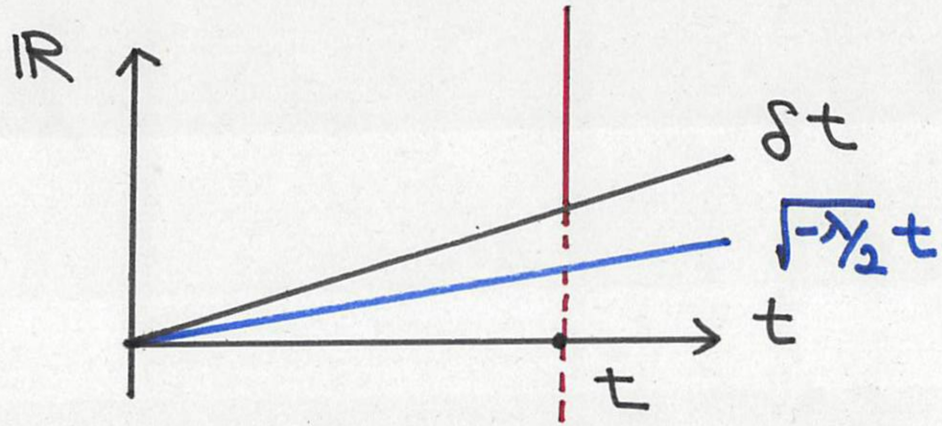

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$\cdot\ \delta \in \left(\sqrt{-\lambda/2}, \sqrt{-2\lambda}\right)$

$$\Rightarrow \mathbb{P}_{\underline{\underline{x}}}(R_t > \delta t) \sim \mathbb{E}_x[Z_t^{\delta t}]$$

$$\sim c_d\, \delta^{\frac{d-1}{2}} \underbrace{e^{(-\lambda - \sqrt{-2\lambda}\,\delta)t}\, t^{\frac{d-1}{2}}}_{\text{exp. decay}} \underline{\underline{h(x)}}$$

[explicit const. including h, λ]

· Chauvin-Rouault (88, 90), McKean (75)

▷ $v_{\mathcal{K}}(t,x) := \mathbb{P}_x(R_t > \mathcal{K})$ $\left(= 1 - \underline{\underline{\mathbb{P}_x(R_t \le \mathcal{K})}}\right)$

$$\Rightarrow \begin{cases} \dfrac{\partial v_{\mathcal{K}}}{\partial t} = \left(\dfrac{1}{2}\Delta + \underline{\mu(1-v_{\mathcal{K}})}\right) v_{\mathcal{K}} \\ v_{\mathcal{K}}(0,x) = 1_{|x| > \mathcal{K}} \end{cases}$$

potential

F-KPP eq.

$$\rightsquigarrow \mathbb{P}_x(R_t > \delta t) = v_{\delta t}(t,x) \qquad \to 0 \ (t \to \infty)$$

$$= E_x\left[e^{\int_0^t (1 - \boxed{v_{\delta t}(t-s, B_s)}) dA_s^{\mu}} ; |B_t| > \delta t \right]$$

$$\sim E_x\left[e^{A_t^{\mu}} ; |B_t| > \delta t \right] = \mathbb{E}_x\left[Z_t^{\delta t} \right]$$

Key Prop.

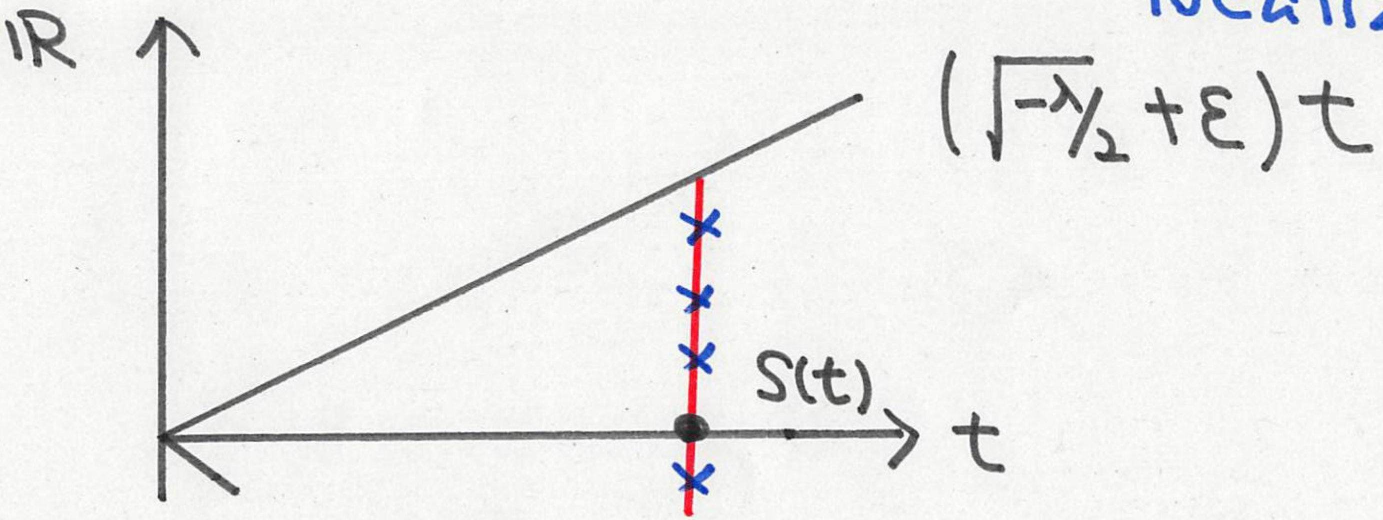
· $g(t) := \underline{\sqrt{-\lambda/2}\, t + \frac{d-1}{2\sqrt{-2\lambda}} \log t + \mathcal{K}}$

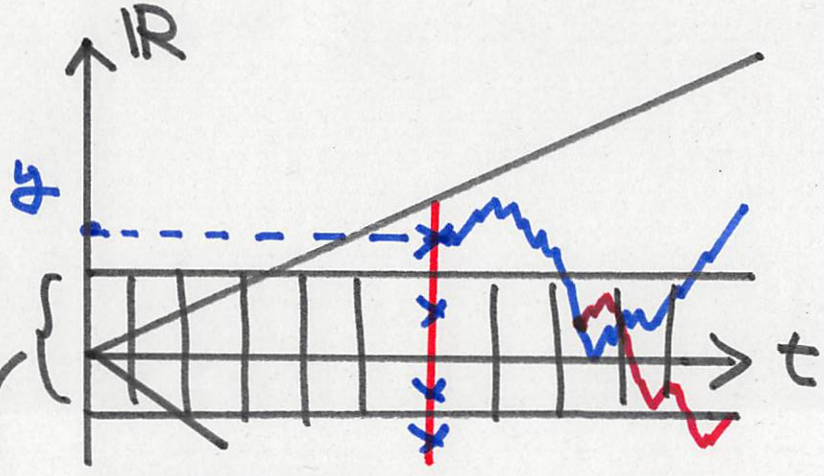
$\Rightarrow \mathbb{P}_x (Y_t \le \mathcal{K}) = \mathbb{P}_x (R_t \le g(t))$

$$\asymp \mathbb{E}_x \Big[\prod_{k=1}^{Z_{s(t)}} \mathbb{P}_{B^k_{s(t)}} (R_{t-s(t)} \le g(t)) ;$$

$$\left[\begin{array}{l} R_t / t \to \sqrt{-\lambda/2} \text{ a.s.} \\ s(t) \to \infty \end{array} \right] \qquad R_{s(t)} \le \left(\sqrt{-\frac{\lambda}{2}} + \varepsilon \right) s(t) \Big]$$

"localization"





Supp of μ

key prop + "localization" on supp μ

$$\mathbb{P}_y\left(L_{t-s(t)} \le \delta(t)\right)$$

$$= 1 - \mathbb{P}_y\left(L_{t-s(t)} > \delta(t)\right)$$

$$\geq 1 - \mathbb{E}_y\left[Z^{\delta(t)}_{t-s(t)}\right]$$

$$\approx 1 - \boxed{e^{-\lambda(t-s(t))}\int_{|z|>\delta(t)} h(z)\,dz \cdot h(y)}$$

$$\approx e^{-\square}$$

$$1 - \square \sim e^{-\square}$$

$$(\square \to 0)$$

[Second moment for the upper bound]

$$\Longrightarrow \quad \mathbb{E}_x\Big[\prod_{k=1}^{Z_{S(t)}} \mathbb{P}_{B^k_{S(t)}}\big(R_{t-S(t)} \le g(t)\big);$$

$$\left[\begin{array}{l} S(t) \\ \simeq \begin{cases} \log t & (d=1) \\ \log\log t & (d=2) \end{cases}\end{array}\right] \qquad R_{S(t)} \le \left(\sqrt{-\lambda/2} + \varepsilon\right) S(t)\Big]$$

$$\simeq \mathbb{E}_x\Big[\underline{\underline{\prod_{k=1}^{Z_{S(t)}}}} e^{-\boxed{e^{-\lambda(t-S(t))} \cdot \int_{|z|>g(t)} h(z)\,dz \cdot \underline{\underline{h(B^k_{S(t)})}}}};$$

$$R_{S(t)} \le \left(\sqrt{-\lambda/2} + \varepsilon\right) S(t)\Big]$$

$$= \mathbb{E}_x\Big[e^{-e^{-\lambda t}\int_{|z|>g(t)} h(z)\,dz \cdot \underline{\underline{M_{S(t)}}}};\ —\Big]$$

$$\exists \lim_{t\to\infty}$$