Scaling limit of uniform spanning tree in three dimensions

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ongoing work with Omer Angel (UBC), David Croydon (Kyoto University) and Sarai Hernandez Torres (UBC)

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- UST has important connections to several areas:
  - Loop-erased random walk (LERW)
  - Loop soup
  - Conformally invariant scaling limits
  - The Abelian sandpile model
  - Gaussian free field
  - Domino tiling
  - Random cluster model
  - Random interlacements
  - Potential theory
  - Amenability $\cdots$

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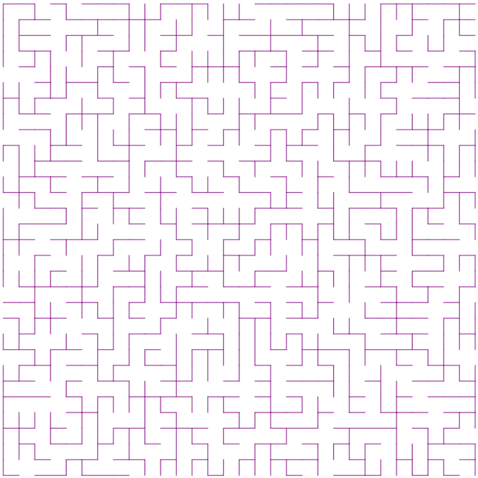
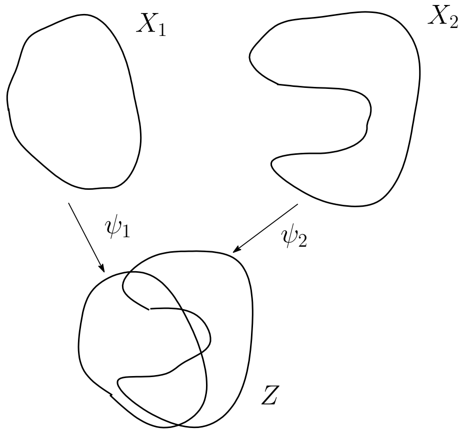


2D UST in a fine grid.
Picture credit: Adrien Kassel.

Uniform Spanning Tree (UST)

Today's talk: Existence of the scaling limit of UST in $2^{-n}\mathbb{Z}^3$ as $n \to \infty$ w.r.t. the Gromov-Hausdorff-Prokhorov topology.

The Gromov-Hausdorff-Prokhorov convergence



Two isometric embeddings $\psi_i : X_i \to Z$ ($i = 1, 2$).

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- For two measured pointed compact metric spaces $\underline{X_i} = (X_i, d_i, \rho_i, \mu_i)$ $(i = 1, 2)$, define $d_{\mathrm{GHP}}(\underline{X_1}, \underline{X_2})$ by

$$d_{\mathrm{GHP}}(\underline{X_1}, \underline{X_2}) = \inf \Big\{ d_Z\Big(\psi_1(\rho_1), \psi_2(\rho_2)\Big) \vee d^Z_{\mathrm{Haus}}\Big(\psi_1(X_1), \psi_2(X_2)\Big) \vee d^Z_{\mathrm{Prokhorov}}\Big(\mu_1 \circ \psi_1^{-1}, \mu_2 \circ \psi_2^{-1}\Big)\Big\},$$

where the infimum is over all metric spaces (Z, d_Z) and isometric embeddings $\psi_i : X_i \to Z$ $(i = 1, 2)$.

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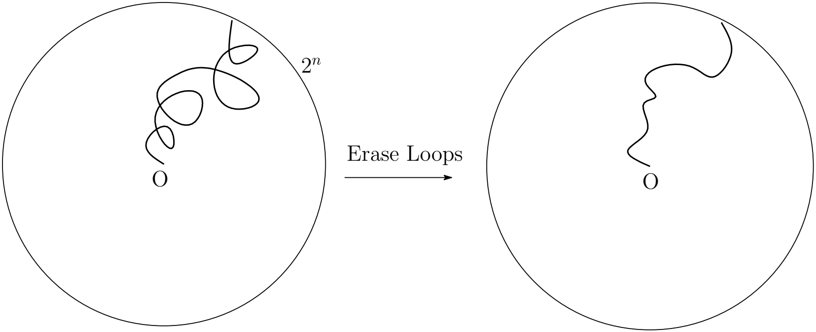
- Let LERW_n be the loop-erased random walk from 0 to $\partial B(2^n)$. Denote the number of steps of LERW_n by $\left|\mathrm{LERW}_n\right|$.

SRW and LERW



SRW (left) and LERW$_n$ (right).

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- (S. '14, Li-S. '18) It is proved that there exists a constant $\beta \in (1, \frac{5}{3}]$ s.t.

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Theorem (Angel-Croydon-S.-Hernandez Torres. '19+)

As $n \to \infty$, the measured pointed tree $(\mathcal{U}_n, 2^{-\beta n} d_{\mathcal{U}_n}, 0, 2^{-3n}\mu_{\mathcal{U}_n})$ converges weakly w.r.t. the GHP topology.

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- **Remark 3**: Kozma ('07) proved the existence of weak convergence limit of 3D LERW w.r.t. the Hausdorff metric. But the topology he used is weaker than we want.
- **Remark 4**: We also study the SRW on $\mathcal{U}_n$ and its scaling limit in our forthcoming paper.

  (Scaling limit of the SRW on 2D UST was studied in Barlow-Croydon-Kumagai ('17).)

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Tightness follows from the following proposition:

Proposition (Angel-Croydon-S.-Hernandez Torres. '19+)

For every $\epsilon, R > 0$, it follows that

(i) $$\lim_{\lambda\to\infty} \limsup_{\delta\to 0} P\left(\delta^3 \mu_{\mathcal{U}}\left(B_{\mathcal{U}}\left(0, \delta^{-\beta} R\right)\right) > \lambda\right) = 0,$$

(ii) $$\lim_{\eta\to 0} \limsup_{\delta\to 0} P\left(\inf_{x\in B_{\mathcal{U}}\left(0,\delta^{-\beta} R\right)} \delta^3 \mu_{\mathcal{U}}\left(B_{\mathcal{U}}\left(x, \delta^{-\beta}\epsilon\right)\right) < \eta\right) = 0,$$

(iii) $$\lim_{\eta\to 0} \limsup_{\delta\to 0} P\left(\inf_{(x,y)\in A} \delta^{\beta} d_{\mathcal{U}}(x, y) < \eta\right) = 0,$$

where $A = \left\{(x, y) \,\middle|\, x, y \in B_{\mathcal{U}}\left(0, \delta^{-\beta} R\right),\ \delta|x - y| > \epsilon\right\}$.

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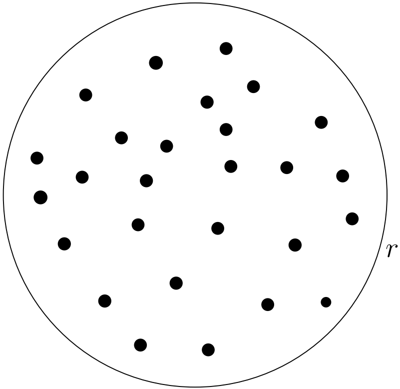

Black points stand for a ϵ-net $\{x_i\}_{i\geq 1}$.

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- Let $\mathcal{U}_n^\epsilon$ be the subtree of $\mathcal{U}_n$ spanned by $\{x_i\}$, i.e.,

$$\mathcal{U}_n^\epsilon = \bigcup_i \gamma_n(x_i).$$

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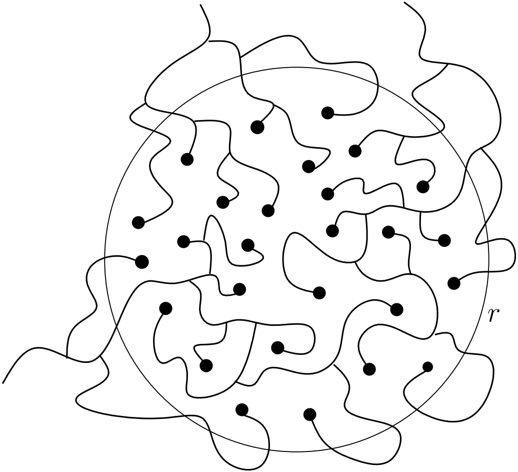

The subtree $\mathcal{U}_n^\epsilon$ is drawn by solid curves.

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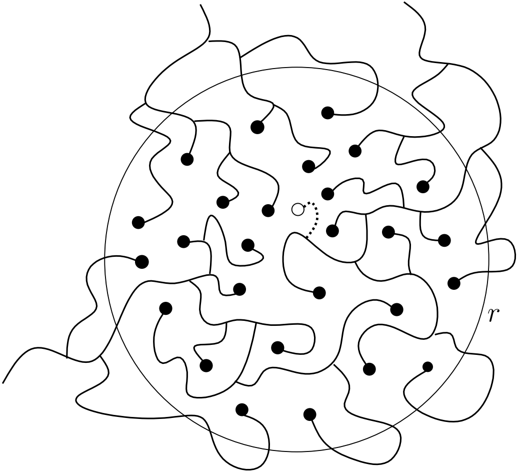

The full tree $\mathcal{U}_n \cap B(r)$ is well approximated by $\mathcal{U}_n^\epsilon \cap B(r)$.

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$\longrightarrow$ The problem boils down to the convergence of $\mathcal{U}_n^\epsilon$ as $n \to \infty$.

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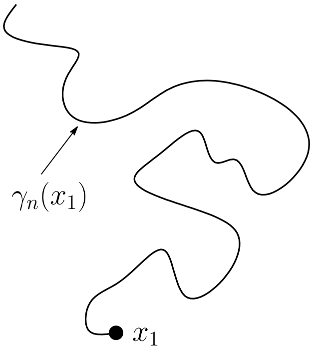

$\gamma_n(x_1) \overset{d}{=}$ (infinite LERW starting from x_1).

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The convergence of the infinite LERW in the natural parametrization is proved as follows.

Theorem (Li-S. '18)

As $n \to \infty$, η_n converges weakly to a random continuous curve w.r.t. the uniform convergence topology.

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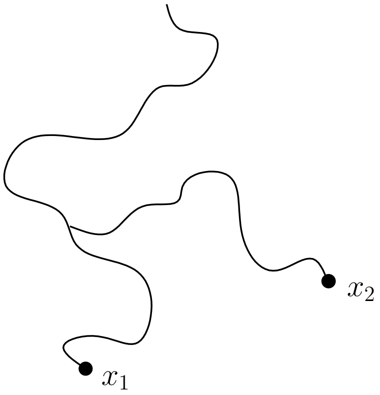

(The second branch) $=$ LE(R^{x_2}),
where R^{x_2} is the SRW starting from x_2 until it hits $\gamma_n(x_1)$.

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- (Angel-Croydon-S.-Hernandez Torres. '19+)
  Proved the convergence in the natural parametrization for LERW in the complement of LERW's.

Open Problem

What is the scaling limit of 3D UST?

Can we give a "nice" description of it?

Thank you for your attention!