Existence and Uniqueness of Quasi-Stationary Distributions for Symmetric Markov Processes with Tightness Property

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2019/9/2

Objective

$X = (\mathrm{P}_x, X_t, \zeta)$: a symmetric Markov process on E.
Suppose X is almost surely killed, i.e.,

$$\mathrm{P}_x(\zeta < \infty) = 1, \quad \forall x \in E.$$

• A probability measure $\mu \in \mathcal{P}(E)$ is said to be a **quasi-stationary distribution (QSD)** if for all $B \in \mathcal{B}(E)$

$$\mu(B) = \mathrm{P}_\mu(X_t \in B \,|\, t < \zeta) = \frac{\mathrm{P}_\mu(X_t \in B)}{\mathrm{P}_\mu(X_t \in E)}.$$

Here $\mathrm{P}_\mu = \int_E \mathrm{P}_x d\mu(x)$.

• Our objective is to show that if X is irreducible, strong (resolvent) Feller and has a tightness property defined below, the existence and uniqueness of QSDs holds.

Setting

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- E : locally compact separable metric space
- m : positive Radon measure with $\mathrm{supp}[m] = E$
- $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$: regular Dirichlet form on $L^2(E; m)$
- $X = (\Omega, X_t, \mathrm{P}_x, \zeta)$: m-symmetric Markov process on E generated by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$

$$(p_t f, g)_m = (f, p_t g)_m, \quad f, g \in \mathcal{B}_b(E).$$

Here $\{p_t\}_{t \geq 0}$ is the semigroup of X,

$$p_t f(x) = \mathrm{E}_x[f(X_t)], \quad f \in \mathcal{B}_b(E).$$

- $\{R_\beta\}_{\beta \geq 0}$: the resolvent of X,

$$R_\beta f(x) = \int_0^\infty e^{-\beta t} p_t f(x) dt, \quad f \in \mathcal{B}_b(E).$$

X is said to be in Class (T) if

(i) (Irreducibility) If a Borel set A is p_t-invariant, i.e., $\int_A p_t 1_{A^c} dm = 0$ for any $t > 0$, then $m(A) = 0$ or $m(A^c) = 0$.

(ii) (Strong Resolvent Feller Property) $R_\beta(\mathcal{B}_b(E)) \subset C_b(E)$, $\beta > 0$.

(iii) (Tightness) For any $\epsilon > 0$, there exists a compact set $K \subset E$ s.t.

$$\sup_{x \in E} R_1 1_{K^c}(x) \leq \epsilon.$$

- If X is conservative, $\{R_1(x, \bullet)\}_{x \in E}$ is tight in Prohorov's sense.
- $R_1 1 \in C_\infty(E)$ (explosive) $\Longrightarrow$ (iii). Here $1 = 1_E$. Indeed

$$\sup_{x \in E} R_1 1_{K^c}(x) = \sup_{x \in K^c} R_1 1_{K^c}(x) \leq \sup_{x \in K^c} R_1 1(x).$$

• If Khasminskii's test confirms the explosion of a symmetric diffusion process X, then X is considered to have the tightness property.

• If $m(E) < \infty$ and $\|p_t\|_{1,\infty} = c_t < \infty$, then (iii) holds. For example, an absorbing BM on a domain with finite volume.

Fact 1 (One-dimensional diffusion process)

A one-dimensional minimal diffusion has the tightness property if and only if there exists no natural boundary.

• Fact 1 follows from asymptotic properties of 1-resolvent near boundary (e.g. Itô, K.: Essentials of stochastic processes, American Mathematical Society).

We give some properties of symmetric Markov processes in Class (T).

Theorem

If X is in Class (T), then $(\mathcal{D}(\mathcal{E}), \mathcal{E}_1(= \mathcal{E} + (\ ,\ )_m))$ is compactly embedded in $L^2(E; m)$($\Longleftrightarrow p_t$ is a compact operator on $L^2(E; m)$).

Theorem

If X in Class(T) is not conservative, it explodes very fast. In fact, the life time ζ is exponentially integrable:

$$\sup_{x \in E} \mathbf{E}_x[e^{\gamma\zeta}] < \infty \Longleftrightarrow \gamma < \lambda_0,$$

$$\lambda_0 = \inf\{\mathcal{E}(u, u) \mid u \in \mathcal{D}(\mathcal{E}),\ \|u\|_2 = 1\} = \mathcal{E}(\phi_0, \phi_0) > 0.$$

Theorem

If X is in Class (T), every L^2-eigenfunction of p_t has a bounded continuous version.

• The boundedness follows from the next inequality: for a large n

$$|\phi(x)| \le \sup_{x\in K_n} |\phi(x)| \cdot \sup_{x\in E} \mathrm{E}_x\left[e^{\lambda \tau_{K_n^c}}\right] < \infty.$$

Here $\phi \in C(\{K_n\})$ be the eigenfunction corresponding to the eigenvalue λ and $\tau_{K_n^c} = \inf\{t > 0 \mid X_t \notin K_n^c\}$. More precisely, let $\lambda_{K_n^c}$ be the principal eigenvalue on K_n^c with Dirichlet boundary condition. By Tightness (iii)

$$\lambda_{K_n^c} \uparrow \infty, \quad n \uparrow \infty.$$

Noting that

$$\sup_{x\in E} \mathrm{E}_x\left[e^{\gamma \tau_{K_n^c}}\right] < \infty \Longleftrightarrow \gamma < \lambda_{K_n^c}.$$

we have $\sup_{x\in E} \mathrm{E}_x\left[e^{\lambda \tau_{K_n^c}}\right] < \infty$ by taking n so large that $\lambda_{K_n^c} > \lambda$.

• The continuity of eigenfunctions follows from the strong Feller property.

- If X in Class (T) is not conservative, then

$$\mathrm{P}_x(\zeta < \infty) = 1, \ \forall x \in E \text{ (almost surely killed).}$$

In the sequel, we assume X is not conservative.

Theorem 1

If X in Class (T) is not conservative, it has a unique QSD with the following expression:

$$\nu^{\phi_0}(B) = \frac{\int_B \phi_0 dm}{\int_E \phi_0 dm}, \quad B \in \mathcal{B}(E).$$

Here ϕ_0 is the principal eigenfunction ($\phi_0 > 0$):

$$\begin{aligned}\lambda_0 &= \inf\{\mathcal{E}(u,u) \mid u \in \mathcal{D}(\mathcal{E}), \ \|u\|_2 = 1\} \\ &= \mathcal{E}(\phi_0, \phi_0) > 0.\end{aligned}$$

- For the definition of ν^{ϕ_0}, it is necessary that $\phi_0 \in L^1(E, m)$. This fact is not trivial because $m(E)$ can be infinite.

Key Lemma

If X is in Class (T), then the normalized principal eigenfunction ϕ_0 belongs to $L^1(E, m)$.

(Sketch of Proof). Fix a compact set K with $m(K) > 0$. Define

$$p_t^K f(x) = \mathbf{E}_x\left[e^{-\int_0^t 1_K(X_s)ds} f(X_t)\right],$$

$$p_t^{\lambda_0, K} f = e^{\lambda_0 t} p_t^K f, \quad R^{\lambda_0, K} f = \int_0^\infty p_t^{\lambda_0, K} f.$$

$$\left(e^{\lambda_0 t} p_t \phi_0 = \phi_0, \quad \int_0^\infty e^{\lambda_0 t} p_t \phi_0 dt = \int_0^\infty \phi_0 dt = \infty.\right)$$

Note that

$$\phi_0(x) = R^{\lambda_0, K}(1_K \phi_0)(x), \ \forall x \in E.$$

$$-(A+\lambda_0-1_K)\phi_0=-(A+\lambda_0)\phi_0+1_K\phi_0=1_K\phi_0.$$

$$R^{\lambda_0,K}(1_K\phi_0)=R^{\lambda_0,K}(-(A+\lambda_0-1_K)\phi_0)=\phi_0.$$

By the symmetry of $R^{\lambda_0,K}$ with respect to m

$$\int_E \phi_0 dm=\int_E R^{\lambda_0,K}(1_K\phi_0)dm=\int_E 1_K\phi_0 R^{\lambda_0,K}1dm$$
$$\le \|R^{\lambda_0,K}1\|_\infty \int_K \phi_0 dm<\infty$$

if $\|R^{\lambda_0,K}1\|_\infty<\infty$.

• We can prove $\|R^{\lambda_0,K}1\|_\infty<\infty$ using L^p-independence of the growth bound of p_t^K:

$$\lim_{t\to\infty}\frac{1}{t}\log\|p_t^K\|_{\infty,\infty}=\lim_{t\to\infty}\frac{1}{t}\log\|p_t^K\|_{2,2}<-\lambda_0$$

(Chen-Kim-Kuwae)

Let us prove the main theorem:

Theorem 1

If X in Class (T) is not conservative, then

$$\nu^{\phi_0}(B) = \frac{\int_B \phi_0 dm}{\int_E \phi_0 dm}, \quad B \in \mathcal{B}(E)$$

is a unique QSD.

Let $X^{\phi_0} = (\Omega, X_t, \mathrm{P}_x^{\phi_0})$ be the transformed process by

$$\mathrm{P}_x^{\phi_0} = L_t^{\phi_0} \cdot \mathrm{P}_x, \quad L_t^{\phi_0} = e^{\lambda_0 t} \frac{\phi_0(X_t)}{\phi_0(X_0)} 1_{\{t<\zeta\}}$$

(ground state transform).

- **X^{ϕ_0} is $\phi_0^2 m$-symmetric, conservative, irreducible process. Its semigroup is expressed as**

$$p_t^{\phi_0} f(x) = e^{\lambda_0 t} \frac{1}{\phi_0(x)} p_t(\phi_0 f)(x)$$

$$\Longleftrightarrow \quad p_t f(x) = \exp(-\lambda_0 t)\phi_0(x) p_t^{\phi_0}(f/\phi_0)(x).$$

Existence of QSD

Claim 1

$$\nu^{\phi_0}(B) := \frac{\int_B \phi_0 dm}{\int_E \phi_0 dm}, \quad B \in \mathcal{B}(E)$$

is a QSD, i.e., $\mathrm{P}_{\nu^{\phi_0}}(X_t \in B \,|\, t < \zeta) = \nu^{\phi_0}(B)$.

$$\nu^{\phi_0} : \mathrm{QED} \iff \mathrm{P}_{\nu^{\phi_0}}(X_t \in B | t < \zeta) = \frac{\mathrm{P}_{\nu^{\phi_0}}(X_t \in B)}{\mathrm{P}_{\nu^{\phi_0}}(X_t \in E)}.$$

By the definition of ν^{ϕ_0}

$$\mathrm{P}_{\nu^{\phi_0}}(X_t \in B) = \frac{\int_E \mathrm{P}_x(X_t \in B)\phi_0(x)dm}{\int_E \phi_0(x)dm}.$$

$$\begin{aligned}\int_E \mathrm{P}_x(X_t \in B)\phi_0(x)dm &= \int_E p_t 1_B(x)\phi_0(x)dm \\ &= e^{-\lambda_0 t}\int_E p_t^{\phi_0}\left(\frac{1_B}{\phi_0}\right)\phi_0^2 dm.\end{aligned}$$

By the symmetry of $p_t^{\phi_0}$ with respect to $\phi_0^2 m$, the RHS. equals

$$e^{-\lambda_0 t}\int_E \left(\frac{1_B}{\phi_0}\right) p_t^{\phi_0} 1 \cdot \phi_0^2 dm = e^{-\lambda_0 t}\int_B \phi_0 \, dm,$$

and thus

$$\mathrm{P}_{\nu^{\phi_0}}(X_t \in B) = \frac{e^{-\lambda_0 t}\int_B \phi_0 \, dm}{\int_E \phi_0(x) dm}.$$

Hence

$$\begin{aligned}\mathrm{P}_{\nu^{\phi_0}}(X_t \in B \,|\, t < \zeta) &= \frac{e^{-\lambda_0 t}\int_B \phi_0 \, dm / \int_E \phi_0 \, dm}{e^{-\lambda_0 t}\int_E \phi_0 \, dm / \int_E \phi_0 \, dm} \\ &= \nu^{\phi_0}(B),\end{aligned}$$

which implies that ν^{ϕ_0} is a QSD.

Uniquness of QSD

Suppose μ is a QSD. By definition of QSD

$$\mu(B) = \frac{\mathrm{P}_\mu (X_t \in B)}{\mathrm{P}_\mu (X_t \in E)}, \ \forall B \in \mathcal{B}(E).$$

For compact sets $K, F \subset E$

$$\mu(K) = \frac{\mathrm{P}_\mu (X_t \in K)}{\mathrm{P}_\mu (X_t \in E)} \leq \frac{\mathrm{P}_\mu (X_t \in K)}{\mathrm{P}_\mu (X_t \in F)}.$$

Since

$$\mathrm{P}_x (X_t \in K) = p_t 1_K(x) = e^{-\lambda_0 t}\phi_0(x) p_t^{\phi_0}(1_K/\phi_0)(x),$$

$$(RHS) = \frac{e^{-\lambda_0 t}\int_E \phi_0 p_t^{\phi_0}(1_K/\phi) d\mu}{e^{-\lambda_0 t}\int_E \phi_0 p_t^{\phi_0}(1_F/\phi) d\mu}$$

$$= \frac{\int_E \phi_0 p_t^{\phi_0}(1_K/\phi) d\mu}{\int_E \phi_0 p_t^{\phi_0}(1_F/\phi) d\mu}.$$

Since

$$\sup_{x \in E} \frac{1_K}{\phi_0}(x) = \frac{1}{\inf_{x \in K} \phi_0(x)} < \infty,$$

$1_K/\phi_0 \in L^\infty(E; \phi_0^2 m)$. Similarly, $1_F/\phi_0 \in L^\infty(E; \phi_0^2 m)$.

$$\lim_{t \to \infty} p_t^{\phi_0} \left(\frac{1_K}{\phi_0} \right)(x) = \int_E \frac{1_K}{\phi_0} \phi_0^2 dm = \int_K \phi_0 dm$$

by applying Fukushima's ergodic theorem to X^{ϕ_0}.

Theorem (Fukushima's ergodic theorem)

Let X be an irreducible m-symmetric Markov process with transition probability density, $p_t(x, dy) = p_t(x, y)m(dy)$. Assume that $m(E) < \infty$. Then for $f \in L^\infty(E; m)$

$$\lim_{t \to \infty} p_t f(x) = \frac{\int_E f dm}{m(E)}, \quad \forall x \in E.$$

- X^{ϕ_0} satisfies the conditions in Theorem above.

By the bounded convergence theorem

$$\lim_{t\to\infty}\int_E \phi_0 p_t^{\phi_0}\left(\frac{1_K}{\phi_0}\right)d\mu = \int_E \phi_0 d\mu \int_K \phi_0 dm.$$

$$\mu(K) \le \liminf_{t\to\infty} \frac{\int_E \phi_0 p_t^{\phi_0}(1_K/\phi)d\mu}{\int_E \phi_0 p_t^{\phi_0}(1_F/\phi)d\mu}$$
$$= \frac{\int_K \phi_0 dm}{\int_F \phi_0 dm} \downarrow \nu^{\phi_0}(K),\ \ F\uparrow E.$$

Hence $\mu(K) \le \nu^{\phi_0}(K)$ and so

$$\mu(B) \le \nu^{\phi_0}(B),\ \forall B \in \mathcal{B}(E).$$

Noting that $\mu(B) = 1-\mu(B^c) \ge 1-\nu^{\phi_0}(B^c) = \nu^{\phi_0}(B)$, we have

$$\mu = \nu^{\phi_0}.$$

- If $\mu \in \mathcal{P}(E)$ satisfies

$$\lim_{t\to\infty} \mathrm{P}_x(X_t \in B \mid t < \zeta) = \mu(B), \ \forall x \in E, \ \forall A \in \mathcal{B}(E),$$

μ is said to be **Yaglom limit.**

- p_t is said to be **intrinsically ultracontractive** (abreviated as IU), if

$$\|p_t^{\phi_0}\|_{1,\infty} \le C(t) < \infty, \quad t > 0,$$

where $\|\cdot\|_{1,\infty}$ is the operator norm from $L^1(E;\phi_0^2 m)$ to $L^\infty(E;\phi_0^2 m)$.

Theorem 2

X is in Class (T). In addition, if its semigroup p_t is intrinsically ultracontractive, then for any initial distribution $\nu \in \mathcal{P}(E)$

$$\lim_{t\to\infty} \mathrm{P}_\nu(X_t \in B \mid t < \zeta) = \nu^{\phi_0}(B), \ \forall B \in \mathcal{B}(E),$$

that is, ν^{ϕ_0} is the Yaglom limit.

Examples

Example 1

A one-dimentional diffusion process without natural boundary has a unque QSD.

- **A sufficient condition for the IU is given in terms of speed measure m and scale function s (Tomisaki).**

Example 2

Let (P_x, X_t) be the symmetric α-stable process on $\mathbb{R}^d$. If an open set $D \subset \mathbb{R}^d$ satisfies

$$\lim_{|x|\to\infty} m(D \cap B(x,1)) = 0 \quad (\text{thin at infinity}),$$

then the absorbing symmetric α-stable process on D is in Class (T), consequently, QSD is unique. Here m is the Lebesgue measure and $B(x,1)$ is the open ball with center x and radius 1.

• For any bounded open set D, the semigroup of the absorbing symmetric α-stable process on D is IU (T. Kulczycki, 1998). As a result,

$$\lim_{t \to \infty} \mathrm{P}_x^D \left(X_t \in B \,|\, t < \tau_D \right) = \nu^{\phi_0}(B).$$

For absorbing Brownian motion, this result is due to R. Pinsky. In this case, the smoothness of the boundary ∂D is necessary.

• There exists an example of an open set with infinite volume on which the absorbing symmetric α-stable process satisfies the IU (M. Kwaśnicki, 2009).

Example 3

Let X be the symmetric α-stable process on $\mathbb{R}^d$ and V a locally bounded, non-negative function on $\mathbb{R}^d$ satisfying

$$\lim_{|x|\to\infty} \frac{V(x)}{\log|x|} = \infty.$$

Define

$$\mathrm{P}_x^V = e^{-\int_0^t V(X_s)ds} \cdot \mathrm{P}_x \quad \text{on } \mathcal{F}_t.$$

Then P_x^V is in Class (T) and its semigroup is intrinsically ultracontractive (Kaleta, K., Kulczycki, T.). Let ϕ_0 be the principal eigenfunction of $(-\Delta)^{\alpha/2} + V$. Then

$$\mathrm{P}_x^V(X_t \in A \,|\, t < \zeta) = \frac{\mathrm{E}_x\left[\exp(-\int_0^t V(X_s)ds); X_t \in A\right]}{\mathrm{E}_x\left[\exp(-\int_0^t V(X_s)ds)\right]} \longrightarrow \nu^{\phi_0}(A), \quad t \to \infty.$$