On beta Laguerre ensembles at varying temperature

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Wishart matrices

- $G = \{g_{ij}\}_{M\times N}$, where g_{ij}: i.i.d. standard real Gaussian $\mathcal{N}(0,1)$,

$$X = X(M,N) = \frac{1}{M} G^t G \quad : \text{Wishart matrix.}$$

  - X is an $N \times N$ symmetric, non-negative definite matrix;
  - analog of chi-squared distributions;
  - invariant under orthogonal conjugation;
  - for $M \geq N$, the eigenvalues $(\lambda_1, \lambda_2, \dots, \lambda_N)$ have joint distribution

$$(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_j - \lambda_i| \prod_{i=1}^{N} \left(\lambda_i^{\frac{1}{2}(M-N+1)-1} e^{-\frac{M}{2}\lambda_i} \right), \quad \lambda_i > 0.$$

Marchenko–Pastur law

Histogram of the eigenvalues of $X(M, N)$, $M = 5000$, $N = 1000$,

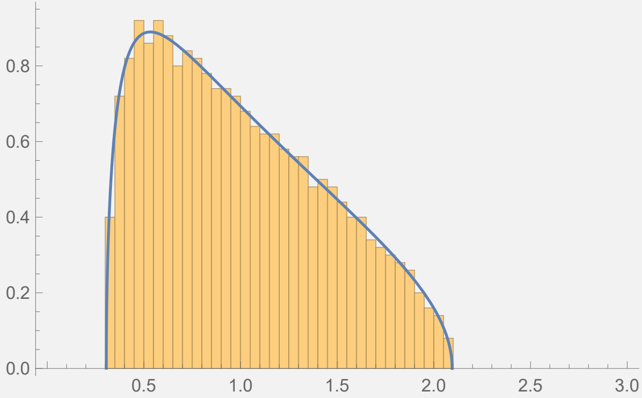

Figure: Marchenko–Pastur distribution with parameter $\gamma = N/M = 0.2$,
$mp_\gamma(x) = \frac{1}{2\pi\gamma x}\sqrt{(\lambda_+ - x)(x - \lambda_-)}, \quad \lambda_\pm = (1 \pm \sqrt{\gamma})^2.$

Marchenko–Pastur law

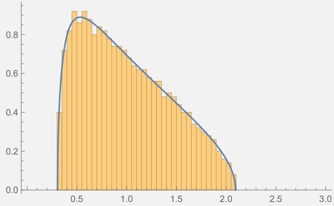

Figure: Histogram of Wishart matrix (5000,1000)

Marchenko–Pastur distribution with parameter γ

$$mp_\gamma(x) = \frac{1}{2\pi\gamma x}\sqrt{(\lambda_+ - x)(x - \lambda_-)},$$

$$\lambda_\pm = (1 \pm \sqrt{\gamma})^2;$$

- As $N \to \infty$ with $N/M \to \gamma$, the empirical distribution

$$L_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$$

converges weakly to mp_γ, a.s.;

- for bdd conti. function $f(x)$, a.s.

$$\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) \to \int f(x) mp_\gamma(dx).$$

Gaussian fluctuations around the limit

- Marchenko–Pastur law: as $N \to \infty$ with $N/M \to \gamma \in (0,1)$,

$$\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) \to \int f(x) mp_\gamma(dx), \quad a.s.,$$

  for bdd conti. function $f(x)$.

- For nice function f, (see the book of Pastur–Shcherbina 2011)

$$\left(\sum_{i=1}^{N} f(\lambda_i) - \mathbf{E}[\cdot]\right) \overset{d}{\to} \mathcal{N}(0, \sigma_f^2),$$

  where

$$\sigma_f^2 = \frac{1}{2\pi^2}\int_{\lambda_-}^{\lambda^+}\int_{\lambda_-}^{\lambda^+}\left(\frac{f(y)-f(x)}{y-x}\right)^2 \frac{4\gamma - (x-\lambda_m)(y-\lambda_m)}{\sqrt{4\lambda - (x-\lambda_m)^2}\sqrt{4\gamma-(y-\lambda_m)^2}} dxdy,$$

  $\lambda_m = (\lambda_- + \lambda^+)/2 = 1 + \gamma$.

! Note that $\mathbf{E}[\cdot]$ can be replaced by $\int f(x) mp_\gamma(dx)$+something.

How to prove those results

- for $f(x) = x^k, k = 1, 2, \dots,$ or $f =$polynomial,

$$\sum_{i=1}^{N} f(\lambda_i) = \mathrm{trace}\, f(X).$$

  Recall that $X = \frac{1}{M} G^t G$. Then combinatorial arguments work.
- For the Marchenko–Pastur law, polynomial test functions are enough.
- For Gaussian fluctuations, we need a type of Poincaré inequality.
- Analysis based on the joint density also works.

- Beta Laguerre ensembles

$$(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_j - \lambda_i|^\beta \prod_{i=1}^{N} \left(\lambda_i^{\frac{\beta}{2}(M-N+1)-1} e^{-\frac{\beta M}{2}\lambda_i} \right), \quad \lambda_i > 0.$$

  - Laguerre matrices ($\beta = 2$) are the complex version of Wishart matrices.
  - The above has meaning for any $\beta > 0$ and any $M > N - 1$.
  - β: inverse temperature

$$(\lambda_1, \dots, \lambda_N) \propto \exp\left(-\beta \left(\sum_{i \neq j} W(\lambda_i, \lambda_j) - N \sum_i V(\lambda_i) \right) \right).$$

- Do the Marchenko–Pastur law and Gaussian fluctuations still hold?
  - YES. An approach related to potential theory (Johansson 1998).
  - A random tridiagonal matrix model for βLE was introduced (Dumitriu and Edelman 2002). Then combinatorial arguments work.
- What happens when β varies? This is the main aim of this talk.

Random Jacobi matrix model for βLE

- $J_N = B_N(B_N)^t$: symmetric, tridiagonal (called Jacobi matrix), where

$$B_N = \frac{1}{\sqrt{\beta M}} \begin{pmatrix} \chi_{\beta M} & & & \\ \chi_{\beta(N-1)} & \chi_{\beta(M-1)} & & \\ & \ddots & \ddots & \\ & & \chi_{\beta} & \chi_{\beta(M-N+1)} \end{pmatrix},$$

  for $\beta > 0, M > N-1$. Then the eigenvalues of J_N are distributed as βLE

$$(\lambda_1, \dots, \lambda_N) \propto \prod_{i<j} |\lambda_j - \lambda_i|^\beta \prod_{i=1}^{N} \left(\lambda_i^{\frac{\beta}{2}(M-N+1)-1} e^{-\frac{\beta M}{2}\lambda_i} \right), \quad \lambda_i > 0.$$

- Why? Based on tridiagonalizing Wishart matrices or Laguerre matrices.

Jacobi matrices

μ: nontrivial prob. meas. on $\mathbb{R}$ s.t. $\int |x|^k d\mu(x) < \infty, k = 0, 1, \dots$.

- $\{1, x, x^2, \dots\}$ are independent in $L^2(\mathbb{R}, \mu)$.
- Define $\{P_n(x)\}_{n=0}^\infty$ as $\begin{cases} P_n(x) = x^n + \text{lower order}, \\ P_n \perp x^j, \quad j = 0, \dots, n-1. \end{cases}$
- $p_n := P_n / \|P_n\|_{L^2}$.

Theorem

(i) $xp_n(x) = b_{n+1} p_{n+1}(x) + a_{n+1} p_n(x) + b_n p_{n-1}(x), \quad n = 0, 1, \dots,$
where $b_{n+1} = \frac{\|P_n\|}{\|P_{n+1}\|}, a_{n+1} = \frac{\langle P_n, xP_n \rangle}{\|P_n\|^2}, P_{-1} \equiv 0.$

(ii) *Multiplication by x in the orthonormal set $\{p_j\}$ has the matrix*

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad Jp = xp, p = (p_0, p_1, \dots,)^t.$$

* *The matrix J is called the Jacobi matrix of the probability measure μ.*

Spectral measures of Jacobi matrices

- Given a Jacobi matrix J, finite or infinite

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad a_i \in \mathbb{R}, b_i > 0.$$

- There is a measure μ on $\mathbb{R}$ s.t.

$$\langle \mu, x^k \rangle = (J^k e_1, e_1) = J^k(1,1), \quad k = 0, 1, \dots$$

- In case of uniqueness, μ is called the spectral measure of J. Uniqueness is equivalent to the essential self-adjointness of J on $\ell^2(\mathbb{N})$.

Some examples of Jacobi matrices

- Semicircle distribution

$$sc(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{1}_{[-2,2]}(x), \leftrightarrow J = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

- Standard Gaussian distribution

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \leftrightarrow J = \begin{pmatrix} 0 & \sqrt{1} & & & \\ \sqrt{1} & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

- Gamma distributions, or Laguerre weights $\frac{1}{\Gamma(\alpha+1)}x^\alpha e^{-x}\mathbf{1}_{(0,\infty)}(x), (\alpha > -1)$

$$J = \begin{pmatrix} \sqrt{\alpha+1} & & & \\ \sqrt{1} & \sqrt{\alpha+2} & & \\ & \sqrt{2} & \sqrt{\alpha+3} & \\ & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \sqrt{\alpha+1} & \sqrt{1} & & \\ & \sqrt{\alpha+2} & \sqrt{2} & \\ & & \sqrt{\alpha+3} & \sqrt{3} \\ & & & \ddots & \ddots \end{pmatrix}.$$

Finite Jacobi matrices

- μ: trivial probability measure, i.e.,

$$\mu = \sum_{j=1}^{N} q_j^2 \delta_{\lambda_j}, \quad \begin{cases} \{\lambda_j\} : \text{distinct}, \\ \sum q_j^2 = 1, q_j > 0. \end{cases}$$

- $\{x^j\}_{j=0}^{N-1}$: independent in $L^2(\mathbb{R}, \mu)$. Define $P_0, \dots, P_{N-1}$.
  $p_n := P_n / \|P_n\|$;
-

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \\ & & b_{N-1} & a_N \end{pmatrix}$$

- $\{\lambda_j\}_{j=1}^{N}$: the eigenvalues of J, $\{v_j\}_{j=1}^{N}$: the corresponding normalized eigenvectors. Then

$$\mu = \sum_{j=1}^{N} |v_j(1)|^2 \delta_{\lambda_j} = \sum_{j=1}^{N} q_j^2 \delta_{\lambda_j}.$$

- $J_N = B_N (B_N)^t, \quad B_N = \frac{1}{\sqrt{\beta M}} \begin{pmatrix} \chi_{\beta M} & & & \\ \chi_{\beta(N-1)} & \chi_{\beta(M-1)} & & \\ & \ddots & \ddots & \\ & & \chi_{\beta} & \chi_{\beta(M-N+1)} \end{pmatrix},$
- The eigenvalues of J_N are distributed as βLE.
- J_N is $1-1$ correspondence with the spectral measure

$$\mu_N = \sum_{j=1}^{N} q_j^2 \delta_{\lambda_j},$$

$(q_1, \dots, q_N)$ is distributed as $(\chi_\beta, \dots, \chi_\beta)$ normalized to unit length, independent of $(\lambda_1, \dots, \lambda_N)$.

Limiting behaviours of βLE

- The spectral measure and the empirical distribution have the same mean

$$\mathbf{E}\Big[\int f d\mu_N\Big] = \mathbf{E}\Big[\sum q_j^2 f(\lambda_j)\Big] = \sum \mathbf{E}[q_j^2]\mathbf{E}[f(\lambda_j)]$$
$$= \frac{1}{N}\sum \mathbf{E}[f(\lambda_j)] = \mathbf{E}\Big[\int f dL_N\Big].$$

or,

$$\mathbf{E}\left[\frac{1}{N}\mathrm{trace}\, f(J_N)\right] = \mathbf{E}[f(J_N)(1,1)].$$

- The limiting behavior of spectral measures follows directly from those of the entries ($N/M \to \gamma \in (0,1)$)

$$B_N = \frac{1}{\sqrt{\beta M}}\begin{pmatrix} \chi_{\beta M} & & & \\ \chi_{\beta(N-1)} & \chi_{\beta(M-1)} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{\beta(M-N+1)} \end{pmatrix} \to \begin{cases} \begin{pmatrix} \frac{1}{\sqrt{\gamma}} & 1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} & \text{as } \beta N \to \infty, \\ \frac{\sqrt{\gamma}}{\sqrt{2c}}\begin{pmatrix} \chi_{\frac{2c}{\gamma}} & & \\ \chi_{2c} & \chi_{\frac{2c}{\gamma}} & \\ & \ddots & \ddots & \ddots \end{pmatrix} & \text{as } \beta N \to 2c. \end{cases}$$

This shows the convergence of the spectral measures, and hence, the convergence of the mean of spectral measures.

- βLE at zero temperature ($\beta \to \infty$)

$$\frac{1}{\sqrt{\beta M}} \begin{pmatrix} \chi_{\beta M} & & & \\ \chi_{\beta(N-1)} & \chi_{\beta(M-1)} & & \\ & \ddots & \ddots & \\ & & \chi_\beta & \chi_{\beta(M-N+1)} \end{pmatrix} \to \frac{1}{\sqrt{M}} \begin{pmatrix} \sqrt{M} & & & \\ \sqrt{N-1} & \sqrt{M-1} & & \\ & \ddots & \ddots & \\ & & \sqrt{1} & \sqrt{M-N+1} \end{pmatrix},$$

- Duality relation between β and $4/\beta$... Then the limiting measure in the regime where $\beta N \to 2c$ with $N/M \to \gamma$ is the spectral measure of Let $\nu_{\gamma,c}$ be the spectral measure of the following Jacobi matrix

$$\frac{\gamma}{c} \begin{pmatrix} \sqrt{\frac{c}{\gamma}} & & \\ \sqrt{c+1} & \sqrt{\frac{c}{\gamma}+1} & \\ & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \sqrt{\frac{c}{\gamma}} & \sqrt{c+1} & \\ & \sqrt{\frac{c}{\gamma}+1} & \sqrt{c+2} \\ & & \ddots & \ddots \end{pmatrix}.$$

This is a (scaled) measure of associated Laguerre polynomials (Ismail et al. 1988).

Conclusion

- Consider βLE with varying temperature β. Then the global asymptotic behavior depends on the limit of βN.
  - Marchenko–Pastur law regime: $\beta N \to \infty$, analog results as in the case beta is fixed;
  - High temperature regime $\beta N \to 2c \in (0, \infty)$:

$$(\lambda_1, \dots, \lambda_N) \propto |\Delta(\lambda)|^{\frac{2c}{N}} \prod_i \lambda_i^{\alpha} e^{-\lambda_i},$$

    with fixed Laguerre weights (fixed α). Then the limiting measure is an associated version of the weight.
  - Analogous results for Gaussian beta ensembles was known (Allez et al. 2012, Shirai–T. 2015, Benaych-Georges–Péché 2015, Nakano–T. 2018, T. 2019).
  - For beta Jacobi ensembles and for general beta ensembles (Liu–Wu 2019, Nakano–T. in preparation).
  - A dynamic version: Wishart beta processes (on going work with Sergio).

Thank you very much for your attention!