Well-posedness for a class of degenerate Itô-SDEs with fully discontinuous coefficients

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Introduction

For $d \geq 2$, consider the stochastic differential equation (SDE)

$$X_t = x_0 + \int_0^t \sqrt{C}(X_s)dW_s + \int_0^t H(X_s)ds, \; t \geq 0, \; x_0 \in \mathbb{R}^d, \qquad (\star)$$

on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_{x_0})$ with

$C = (c_{ij})_{1\leq i,j\leq d}$, symmetric matrix of **bounded** measurable functions,

such that for some $\lambda \geq 1$

$$\lambda^{-1}\|\xi\|^2 \leq \langle C(x)\xi, \xi\rangle \leq \lambda\|\xi\|^2, \quad \text{for all } x, \xi \in \mathbb{R}^d,$$

and

$H = (h_1, ..., h_d)$ vector of **locally bounded** measurable functions.

Uniqueness in law if: for any $x_0 \in \mathbb{R}^d$, the **distribution** $\mathbb{P}_{x_0} \circ X^{-1}$ **is always the same** no matter on which filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_{x_0})$ with X_t $(\mathcal{F}_t)$-adapted and W_t standard $(\mathcal{F}_t)$-BM starting at zero it is considered.

(Stroock/Varadhan 1979)

- if the c_{ij} are continuous and the h_i bounded then $(\star)$ is well-posed, i.e. there exists a solution and it is unique in law.

(Krylov 1969)

- if the h_i are bounded, then $(\star)$ is well-posed for $d = 2$

(Nadirashvili 1997)

- but if $d \geq 3$, there exists an example of a measurable discontinuous C for which uniqueness in law does not hold.

$\rightsquigarrow$ **even in the nondegenerate case,**
well-posedness for discontinuous coefficients is non-trivial

$\rightsquigarrow$ **look for subclasses of discontinuous coefficients,**
in which well-posedness holds

Some subclasses are given when C is **not far from being continuous**

(Krylov 1992, Cerrutti/Escauriaza/Fabes 1993, Safonov 1994)

- if C continuous up to a small set (e.g. a discrete set or a set of α-Hausdorff measure zero with sufficiently small α) then $(\star)$ is well-posed.

or **if set of discontinuities has a special geometric structure**

(Gao 1993)

- if C is continuous up to the common boundary of the upper and lower half spaces then $(\star)$ is well-posed.

(Bass/Pardoux 1997)

- if C is piecewise constant on a decomposition of $\mathbb{R}^d$ into a finite union of polyhedrons then $(\star)$ is well-posed.

A first example with **discontinuous degenerate** C, i.e. merely

$$\langle C(x)\xi, \xi\rangle \geq 0, \qquad \text{for all } x, \xi \in \mathbb{R}^d$$

holds, is given in:

(Krylov 2004)

if for $x = (x_1, y)$, $x_1 \in \mathbb{R}$, $y = (y_1, ..., y_{d-1}) \in \mathbb{R}^{d-1}$ it holds:

- $C(x) = 1_{\{x_1>0\}} C_+(y) + 1_{\{x_1\leq 0\}} C_-(y)$,

  $\mathbf{H}(x) = 1_{\{x_1>0\}} \mathbf{H}_+(y) + 1_{\{x_1\leq 0\}} \mathbf{H}_-(y), \qquad h_1 \equiv 0$
- $\delta \leq C^{11}_{\pm}(y) \leq K$, where $\delta, K \in (0, \infty)$ are some constants
- the first and second order derivatives of $C_\pm$ and $\mathbf{H}_\pm$ are bounded

then $(\star)$ is well-posed.

<u>Goal:</u> **complement the subclasses given in (Bass/Pardoux 1997)** and **(Krylov 2004)** where well-posedness holds in many ways **and find a general class of diffusions with fully discontinuous and degenerate** C **where well-posedness for** $(\star)$ **holds**.

Consider the simplified assumptions:

- $A = (a_{ij})_{1 \le i,j \le d}$ *locally uniformly strictly elliptic* and for some $p > d$

$$a_{ij} \in W^{1,p}_{loc}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$$

  $\psi \in L^{2p}_{loc}(\mathbb{R}^d)$ (resp. $\psi \in L^p_{loc}(\mathbb{R}^d)$) and $\frac{1}{\psi} \in L^\infty_{loc}(\mathbb{R}^d)$

- $\mathbf{G} = (g_1, ..., g_d)$, with $g_i \in L^{2p}_{loc}(\mathbb{R}^d)$ (resp. $g_i \in L^\infty_{loc}(\mathbb{R}^d)$)

$$\Big(\textbf{basically, we need } \ \psi g_i \in L^p_{loc}(\mathbb{R}^d)\Big)$$

Let $\sigma = (\sigma_{ij})_{1 \le i,j \le d} = \sqrt{A}$.

The **corresponding second order linear operator** writes as

$$Lf = \sum_{i,j=1}^{d} \frac{\frac{1}{\psi} a_{ij}}{2} \partial_{ij} f + \sum_{i=1}^{d} g_i \partial_i f, \qquad f \in C_0^\infty(\mathbb{R}^d).$$

Suppose there exists a constant $M > 0$, such that

$$-\frac{\langle A(x)x, x\rangle}{\psi(x)(\|x\|^2+1)} + \frac{\mathrm{trace}(A(x))}{2\psi(x)} + \langle \mathbf{G}(x), x\rangle \leq M\left(\|x\|^2+1\right)\left(\ln(\|x\|^2+1)+1\right)$$

for a.e. $x \in \mathbb{R}^d \setminus K$, K arbitrary compact set.

Then there exists a Hunt process $\mathbb{M} = ((X)_{t\geq 0}, \mathbb{P}_x)_{x\in\mathbb{R}^d})$ that weakly solves

$$X_t = x + \int_0^t \left(\sqrt{\frac{1}{\psi}}\cdot\sigma\right)(X_s)\, dW_s + \int_0^t \mathbf{G}(X_s)\, ds, \quad t \geq 0, \quad x \in \mathbb{R}^d.$$

In this case:

- The **integrals** in the SDE and the solution **do not depend on the Borel version chosen** for $\frac{1}{\psi}$ and **G**.
- The **zeros of** $\frac{1}{\psi}$ **have Lebesgue-Borel measure zero** since $\psi \in L^p_{loc}(\mathbb{R}^d)$
  $\Longrightarrow$ $\sqrt{\frac{1}{\psi}}\cdot\sigma$ may be **degenerate on a set of Lebesgue-Borel measure zero**
- $\sqrt{\frac{1}{\psi}}\cdot\sigma$ and **G** may be **totally discontinuous**
- $\sqrt{\frac{1}{\psi}}\cdot\sigma$ is **locally bounded** since σ is continuous and $\frac{1}{\psi} \in L^\infty_{loc}(\mathbb{R}^d)$, but **G** may be **locally unbounded**

Idea of construction of weak solution:

If it exists, the infinitesimal generator of a solution writes as

$$Lf = \sum_{i,j=1}^{d} \frac{\frac{1}{\psi} a_{ij}}{2} \partial_{ij} f + \sum_{i=1}^{d} g_i \partial_i f, \qquad f \in C_0^{\infty}(\mathbb{R}^d).$$

Theorem (Lee, Trutnau, 2019)

$\exists$ **pre-invariant density** $\rho\psi$ *with* $\rho \in W^{1,p}_{loc}(\mathbb{R}^d) \cap C^{1-d/p}_{loc}(\mathbb{R}^d)$, $\rho > 0$ *pointwise, such that*

$$\int_{\mathbb{R}^d} Lf \, \underbrace{\rho\psi \, dx}_{=:\mu} = 0, \quad \forall f \in C_0^{\infty}(\mathbb{R}^d).$$

Proof.

By [Lee/Trutnau arXiv 2019] (also **[Bogachev, Röckner, Shaposhnikov, 2012]**), since $\psi g_i \in L^p_{loc}(\mathbb{R}^d)$, $\exists \rho \in W^{1,p}_{loc}(\mathbb{R}^d) \cap C^{1-d/p}_{loc}(\mathbb{R}^d)$ with

$$\int_{\mathbb{R}^d} \Big(\sum_{i,j=1}^{d} \frac{a_{ij}}{2} \partial_{ij} f + \sum_{i=1}^{d} \psi g_i \partial_i f \Big) \rho \, dx = 0, \quad \forall f \in C_0^{\infty}(\mathbb{R}^d).$$

$$= \int_{\mathbb{R}^d} \Big(\sum_{i,j=1}^{d} \frac{\frac{1}{\psi} a_{ij}}{2} \partial_{ij} f + \sum_{i=1}^{d} g_i \partial_i f \Big) \rho\psi \, dx$$

□

- This leads by an adaptation of a method of **Stannat 1999** to a **closed extension** $(L_r, D(L_r))$ **of** $(L, C_0^\infty(\mathbb{R}^d))$ **on each** $L^r(\mathbb{R}^d,\mu)$, $r \geq 1$, that **generates a sub-Markovian** C_0**-semigroup of contractions** $(T_t)_{t>0}$ **(and a resolvent** $(G_\alpha)_{\alpha>0}$**)**.

  Heuristics:

$$-\int_{\mathbb{R}^d} Lf\, g\, d\mu = \frac{1}{2}\int_{\mathbb{R}^d} \langle \frac{A}{\psi}\nabla f, \nabla g\rangle \underbrace{\rho\psi dx}_{=\mu} - \int_{\mathbb{R}^d} \langle \frac{1}{\psi}\beta^{\rho,A} - \mathbf{G}, \nabla f\rangle g \underbrace{\rho\psi dx}_{=\mu}$$

("can take ψ in and out")

- adapting arguments of **[Aronson/Serrin, 1967]** with weights in the time derivative term, and of **[Trudinger, 1977]**, **[Gilbarg/Trudinger, 2001]**, **[Bogachev, Krylov, Röckner, Shaposhnikov, 2015]**, we get:

  - $G_\alpha g$, $\alpha > 0$, **has a continuous** μ**-version** $R_\alpha g$, $\quad \forall g \in L^s(\mathbb{R}^d,\mu) + L^\infty(\mathbb{R}^d,\mu)$

    where $s > \frac{d}{2}$ **is such that** $\frac{1}{q} + \frac{1}{s} < \frac{2}{d}$ ($q=$ local integrability order of ψ).

  - $T_t f$, $t > 0$, **has a continuous** μ**-version** $P_t f$, $\quad \forall f \in L^{\frac{2p}{p-2}}(\mathbb{R}^d,\mu) + L^\infty(\mathbb{R}^d,\mu)$

- Then using a result of **Trutnau 2005** and **Shin/Trutnau 2017**, we obtain the weak solution that has the transition function $(P_t)_{t\geq 0}$

- The weak solution satisfies the following **Krylov type estimate for generalized Dirichelt forms:**

  Let $g \in L^r(\mathbb{R}^d, \mu)$ for some $r \in [s, \infty]$ be given. Then for any ball B, there exists a constant $C_{B,r}$, depending in particular on B and r, such that for all $t \geq 0$,

$$\sup_{x\in\overline{B}} \mathbb{E}_x \left[\int_0^t |g|(X_s)\, ds \right] < e^t C_{B,r} \|g\|_{L^r(\mathbb{R}^d,\mu)}. \tag{1}$$

• **Construction method includes existence of a candidate for invariant measure**

(using the Krylov estimate method, we don't have this ...)

if the dual semigroup $(\widehat{T}_t)_{t>0}$ is conservative, then

$$\int_{\mathbb{R}^d} T_t f \, d\mu = \int_{\mathbb{R}^d} f \underbrace{\widehat{T}_t 1_{\mathbb{R}^d}}_{=1} d\mu = \int_{\mathbb{R}^d} f \, d\mu \quad \Longrightarrow \quad \mu \text{ invariant for } (T_t)_{t>0}.$$

In particular

$$\mathbb{P}_\mu(X_t \in A) := \int_{\mathbb{R}^d} \mathbb{P}_x(X_t \in A) \, d\mu = \int_{\mathbb{R}^d} T_t 1_A(x) \, d\mu = \mu(A)$$

$\Longrightarrow \mathbb{P}_\mu$ **stationary "distribution"**

Our results: uniqueness in law

Assume

- $A = (a_{ij})_{1 \le i,j \le d}$ *locally uniformly strictly elliptic* and

  $$a_{ij} \in W^{1,2d+2}_{loc}(\mathbb{R}^d) \cap C(\mathbb{R}^d)$$

  $\psi \in L^q_{loc}(\mathbb{R}^d)$, $q > 2d + 2$, and $\frac{1}{\psi} \in L^\infty_{loc}(\mathbb{R}^d)$.

  Here $\frac{1}{\psi}$ **fixed Borel measurable version** such that

  $$\frac{1}{\psi} \cdot \psi = 1 \text{ a.e.} \quad \text{and} \quad \frac{1}{\psi}(x) \in [0, \infty), \ \forall x \in \mathbb{R}^d$$

- $\mathbf{G} = (g_1, ..., g_d)$, with $g_i \in L^\infty_{loc}(\mathbb{R}^d)$

Let $\sigma = (\sigma_{ij})_{1 \le i,j \le d} = \sqrt{A}$.

Suppose there exists a constant $M > 0$, such that

$$-\frac{\langle A(x)x, x\rangle}{\psi(x)(\|x\|^2 + 1)} + \frac{\mathrm{trace}(A(x))}{2\psi(x)} + \langle \mathbf{G}(x), x\rangle \le M \left(\|x\|^2 + 1\right) \left(\ln(\|x\|^2 + 1) + 1\right)$$

for a.e. $x \in \mathbb{R}^d$.

Theorem (Lee, Trutnau, 2019)

Under the above assumptions, **well-posedness** *holds for the stochastic differential equation*

$$X_t = x + \int_0^t \Big(\sqrt{\frac{1}{\psi}} \cdot \sigma\Big)(X_s)\, dW_s + \int_0^t \mathbf{G}(X_s)\, ds, \quad t \geq 0,\ x \in \mathbb{R}^d. \qquad (\star\star)$$

among all weak solutions $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, X_t = (X_t^1, ..., X_t^d), W = (W^1, \dots, W^m), \mathbb{P}_x)$, $x \in \mathbb{R}^d$, **with**

$$\int_0^\infty 1_{\left\{\sqrt{\frac{1}{\psi}}=0\right\}}(X_s)ds = 0 \qquad \mathbb{P}_x\text{-a.s. } \forall x \in \mathbb{R}^d. \qquad (\star\star\star)$$

Condition $(\star\star\star)$ is satisfied:

- for the solution that we construct
- if $\frac{1}{\psi}(x) \in (0, \infty)$, $\forall x \in \mathbb{R}^d$
- if for each $n \in \mathbb{N}$, $T > 0$ and $x \in \mathbb{R}^d$ it holds

$$\mathbb{E}_x\left[\int_0^T 1_{B_n}\psi(X_s)ds\right] < \infty,$$

where ψ denotes the extended Borel measurable function defined by

$$\psi(x) := \frac{1}{\frac{1}{\psi}(x)}, \quad \text{if } \frac{1}{\psi}(x) \in (0, \infty), \qquad \psi(x) := \infty, \quad \text{if } \frac{1}{\psi}(x) = 0.$$

Example

Let $\alpha \geq 0$ be such that $\alpha(2d+2) < d$ and

$$\frac{1}{\psi}(x) = \frac{\|x\|^{\alpha}}{\phi(x)}, \qquad 0 < c_B \leq \phi \leq C_B \qquad \text{(locally pointwise on balls } B\text{)}$$

Then $\psi \in L^q_{loc}(\mathbb{R}^d)$ for some $q \in (2d+2, \frac{d}{\alpha})$ and $\frac{1}{\psi} \in L^\infty_{loc}(\mathbb{R}^d)$.
Therefore, **our constructed Hunt process $\mathbb{M}$ is the unique (in law) solution of**

$$X_t = x + \int_0^t \|X_s\|^{\alpha/2} \cdot \frac{\sigma}{\sqrt{\phi}}(X_s)\, dW_s + \int_0^t \mathbf{G}(X_s)\, ds, \quad t \geq 0, x \in \mathbb{R}^d.$$

that satisfies $(\star\star\star)$.

If we **choose** the **following Borel measurable version** of $\|x\|^{\alpha/2}$, namely

$$f_\gamma(x) := \|x\|^{\alpha/2} 1_{\{x\neq 0\}}(x) + \gamma 1_{\{x=0\}}(x), \quad x \in \mathbb{R}^d$$

where γ is an **arbitrary but fixed strictly positive real number**. Then

$$\widetilde{X}_t = x + \int_0^t \frac{f_\gamma \cdot \sigma}{\sqrt{\phi}}(\widetilde{X}_s)\, d\widetilde{W}_s + \int_0^t \mathbf{G}(\widetilde{X}_s)\, ds, \quad t \geq 0, \ x \in \mathbb{R}^d,$$

is **well-posed for any** $\gamma > 0$, since

$$\frac{f_\gamma}{\sqrt{\phi}} \in (0, \infty) \implies (\star\star\star).$$

Steps to show uniqueness in law:

First one shows a **local Krylov type estimate that holds for any weak solution $\widetilde{X}$ to ($\star\star$) that satisfies ($\star\star\star$)**:

Lemma (Lee, Trutnau, 2019)

Let $x \in \mathbb{R}^d$, $T > 0$, $R > 0$ and $f \in L^{2d+2,d+1}(B_R \times (0,T))$. Then there exists a constant $C > 0$ which is independent of f such that

$$\widetilde{\mathbb{E}}_x \left[\int_0^{T \wedge \widetilde{D}_R} f(\widetilde{X}_s, s) ds \right] \leq C \|f\|_{L^{2d+2,d+1}(B_R \times (0,T))},$$

where $\widetilde{D}_R = \inf\{t \geq 0 \mid \widetilde{X}_t \in \mathbb{R}^d \setminus B_R\} \nearrow \infty$.

($\star\star\star$) $\Longrightarrow$ **integrals involving the solution and the solution do not depend on the chosen Borel versions for the coefficients**

Now, the **transition function of the weak solution that we construct has such a nice regularity that a local Itô-formula holds for**

$$g(x,t) := P_{T-t}f(x), \quad f \in C_0^\infty(\mathbb{R}^d),\ T > 0,$$

and any weak solution $\widetilde{X}$ **to** ($\star\star$) **that satisfies** ($\star\star\star$):

Lemma (Lee, Trutnau, 2019)

Let $R, T > 0$. *Then* $\widetilde{\mathbb{P}}_x$*-a.s. for any* $x \in \mathbb{R}^d$,

$$g(\widetilde{X}_{T\wedge \widetilde{D}_R}, T \wedge \widetilde{D}_R) - u(x,0) = \int_0^{T\wedge \widetilde{D}_R} \nabla g(\widetilde{X}_s, s)\widehat{\sigma}(\widetilde{X}_s)d\widetilde{W}_s + \int_0^{T\wedge \widetilde{D}_R} (\partial_t g + Lg)(\widetilde{X}_s, s)ds,$$

where $Lg := \frac{1}{2\psi}\mathrm{trace}(A\nabla^2 g) + \langle \mathbf{G}, \nabla g\rangle$.

Moreover

$$\partial_t g + Lg = 0 \quad \text{a.e.}$$

and therefore taking expectations and letting then $R \nearrow \infty$, we get

$$\widetilde{\mathbb{E}}_x[g(\widetilde{X}_T, T)] = g(x,0) = \widetilde{\mathbb{E}}_x[f(\widetilde{X}_T)]$$

$\Longrightarrow$ the **one dimensional marginals coincide** for any weak solution to ($\star\star$) that satisfies ($\star\star\star$) (spends zero time at $\sqrt{\frac{1}{\psi}}$)

By the method of Stroock/Varadhan **uniqueness in law holds** and **any weak solution to $(\star\star)$ is a strong Markov process**

Once uniqueness in law holds, the **local Krylov type estimate can be improved**

We have $q = 2d + 2 + \varepsilon$ for some $\varepsilon > 0$. For $s = \frac{2}{3}d$, we have

$$\frac{1}{q} + \frac{1}{s} = \frac{1}{2d+2+\varepsilon} + \frac{3}{2d} < \frac{2}{d} \quad \Longrightarrow \quad \text{Krylov type estimate for GDFs holds}$$

Then

$$s_0 := \frac{sq}{q-1} = \frac{2}{3}d \cdot \frac{2d+2+\varepsilon}{2d+1+\varepsilon} \Longrightarrow s_0 = \frac{4}{5}d - \delta \ \text{ for small } \delta > 0.$$

Hence for $g \in L^{s_0}(\mathbb{R}^d)_0$ with $\text{supp}(g) \subset B$, where $B \subset \mathbb{R}^d$ is any ball, we have **by the Krylov estimate for generalized Dirichlet forms for** any $x \in \overline{B}$ and **any weak solution $\widetilde{X}$ to $(\star\star)$**

$$\begin{aligned}
\widetilde{\mathbb{E}}_x\left[\int_0^T g(\widetilde{X}_s)ds\right] &\leq C\left(\int_{\mathbb{R}^d} |g|^s \rho\psi dx\right)^{1/s} \\
&\leq C\|\rho\psi\|_{L^q(B)}^{1/s}\left(\int_B |g|^{\frac{sq}{q-1}} dx\right)^{\frac{q-1}{sq}} \\
&\leq \widetilde{c}\,\|g\|_{L^{s_0}(B)}
\end{aligned}$$