Incompressible limit for weakly asymmetric simple exclusion processes

Kenkichi TSUNODA
joint with Milton Jara and Claudio Landim

Department of Mathematics, Osaka University

3, Sep, 2019

Outline of the talk

1. Overview and Related Works
2. Model and Main Results
3. Sketch of the Proof

1. Overview and Related Works

Overview

- The macroscopic density of the $(\mathrm{WASEP})_n$ evolves according to the nonlinear heat eq. as the system size n grows to infinity (Hydrodynamic limit):

$$\partial_t u \;=\; \nabla \cdot [D(u)\nabla u] + \nabla \cdot [\sigma(u)\mathbf{m}] \;, \qquad (1)$$

  where D, σ are $d \times d$-matrices (Diffusivity and Mobility) and $\mathbf{m} \in \mathbb{R}^d$ is a given vector.

- For small $\varepsilon > 0$, let us consider the first order correction to (1) around a constant profile $\alpha_0 \in (0,1)$:

$$\begin{cases} \partial_t u^\varepsilon \;=\; \nabla \cdot [D(u^\varepsilon)\nabla u^\varepsilon] + \varepsilon^{-1}\nabla \cdot [\sigma(u^\varepsilon)\mathbf{m}] \;, \\ u^\varepsilon(0,\cdot) \;=\; \alpha_0 + \varepsilon v_0 \;, \end{cases} \qquad (2)$$

  for some smooth function v_0.

- The solution u^ε should evolve as $u^\varepsilon_t \sim \alpha_0 + \varepsilon v_t$.
- Indeed, if $\sigma'(\alpha_0) = 0$, the sequence $\{\varepsilon^{-1}(u^\varepsilon - \alpha_0)\}_{\varepsilon>0}$ converges to the solution to the Burgers eq. as $\varepsilon \downarrow 0$ (Incompressible limit):

$$\begin{cases} \partial_t v \;=\; \nabla \cdot [D(\alpha_0)\nabla v] + (1/2)\nabla \cdot [v^2 \sigma''(\alpha_0)\mathbf{m}] \;, \\ v(0,\cdot) \;=\; v_0(\cdot) \;. \end{cases}$$

- Main Result (rough version):
  Taking $\varepsilon = \varepsilon_n \downarrow 0$ $(n \to \infty)$, the correctly scaled density of the $(\mathrm{WASEP})_n^{\varepsilon_n}$ evolves according to the Burgers eq.

Related Works

- Many results on hydrodynamic limits.
- Esposito-Marra-Yau, 94, 96 $\cdots$ Derivation of Burgers equation and Navier-Stokes equation ($d \geq 3$).
- Quastel-Yau, 98 $\cdots$ Large deviations for the incompressible limits ($d = 3$).
- Beltán-Landim, 08 $\cdots$ Derivation of Burgers equation and Navier-Stokes equation in any dimensions but with (meso-scopically) big jumps.
- Jara-Menezes, 19+ $\cdots$ Sharp entropy bound.

2. Model and Main Results

Model

- Each particle moves on the d-dimensional discrete torus $\mathbb{T}_n^d = (\mathbb{Z}/n\mathbb{Z})^d = \{1, 2, \cdots, n\}^d$, $n \in \mathbb{N}$. Let $\mathbb{T}^d$ be the d-dimensional torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d = [0, 1)^d$.
- Denote the number of particles at site $x \in \mathbb{T}_n^d$ at time t by $\eta_t^n(x)$ ($\eta_t^n = \{\eta_t^n : x \in \mathbb{T}_n^d\} \in \{0, 1\}^{\mathbb{T}_n^d}$).
- Some parameters:
  - $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$: a sequence converging to 0.
  - $(c_j)_{j=1}^d$: nonnegative local functions.
  - $\mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_d)$: a vector in $\mathbb{R}^d$.

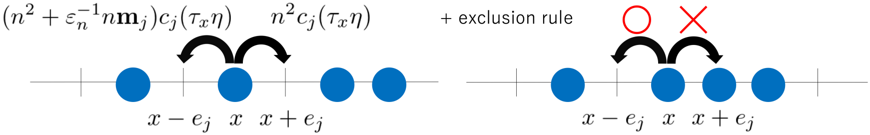
- Let η_t^n be a Markov process on $\{0,1\}^{\mathbb{T}_n^d}$ with the generator $L_n = n^2 L_n^S + \varepsilon_n^{-1} n L_n^A$ with

$$
\begin{aligned}
(L_n^S f)(\eta) &= \sum_{x\in\mathbb{T}_n^d} \sum_{j=1}^{d} c_j(\tau_x\eta) \left\{ f(\sigma^{x,x+e_j}\eta) - f(\eta) \right\} , \\
(L_n^A f)(\eta) &= \sum_{x\in\mathbb{T}_n^d} \sum_{j=1}^{d} \mathbf{m}_j \, c_j(\tau_x\eta) \, \eta_{x+e_j} (1-\eta_x) \\
&\qquad \times \left\{ f(\sigma^{x,x+e_j}\eta) - f(\eta) \right\} .
\end{aligned}
$$

for a function $f : \{0,1\}^{\mathbb{T}_n^d} \to \mathbb{R}$.

The dynamics of our particle system is as follows:

- Each particle can jump from x to $x + e_j$ or $x - e_j$ at given rates only if the site x is occupied and the site $x + e_j$ or $x - e_j$ is vacant.



- For a continuous function $u_0 : \mathbb{T}^d \to [0,1]$, let $\nu^n(u_0)$ be the product Bernoulli measure on $\{0,1\}^{\mathbb{T}_n^d}$:

$$\nu^n(u_0)\,(\eta : \eta(x) = 1) \;=\; u_0(x/n)\;, \quad x \in \mathbb{T}_n^d\;.$$

- Gradient condition: For each j, there exist finitely supported mean zero signed measures $m_{j,p}$, $p = 1, \dots, n_j$ and local functions $g_{j,p}$ such that

$$j_{0,e_j} \;\equiv\; c_j(\eta)[\eta_0 - \eta_{e_j}] \;=\; \sum_{p=1}^{n_j} \sum_{y \in \mathbb{Z}^d} m_{j,p}(y) g_{j,p}(\tau_y \eta)\;.$$

- j_{0,e_j} is called current across the bond $(0, e_j)$.

Classical case ($\varepsilon_n = 1$)

- Assume that $\eta_0^n \overset{\mathrm{d}}{=} \nu^n(u_0)$ for some continuous function $u_0 : \mathbb{T}^d \to [0,1]$.
- Hydrodynamic limit: For any $t \geq 0$ and any smooth function $G : \mathbb{T}^d \to \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}^n \left[\left| \frac{1}{n^d} \sum_{x\in\mathbb{T}_n^d} G(x/n)\eta_t^n(x) - \int_{\mathbb{T}^d} G(x)u(t,x)dx \right| \right] \;=\; 0 \;,$$

where $u : [0,\infty) \times \mathbb{T}^d \to [0,1]$ is the unique weak solution of the Cauchy problem

$$\begin{cases} \partial_t u \;=\; \nabla \cdot [D(u)\nabla u] + \nabla \cdot [\sigma(u)\mathbf{m}] \;, \\ u(0,\cdot) \;=\; u_0(\cdot) \;. \end{cases}$$

Incompressible case ($\varepsilon_n \downarrow 0$)

- Fix $\alpha_0 \in (0,1)$ with $\sigma'(\alpha_0) = 0$ and assume that $\eta_0^n \stackrel{\mathrm{d}}{=} \nu^n(\alpha_0 + \varepsilon_n v_0)$ for some function $v_0 \in C^{3+}(\mathbb{T}^d)$.
- Let $v : [0,\infty) \times \mathbb{T}^d \to \mathbb{R}$ be the unique weak (classical) solution of the Burgers eq.:

$$\begin{cases} \partial_t v \;=\; \nabla \cdot [D(\alpha_0)\nabla v] + (1/2)\nabla \cdot [v^2 \sigma''(\alpha_0)\mathbf{m}] \;, \\ v(0,\cdot) \;=\; v_0(\cdot) \;. \end{cases}$$

- For each $t \geq 0$, let $u_t^n = \alpha_0 + \varepsilon_n v_t, \nu_t^n = \nu^n(u_t^n)$ and let μ_t^n be the distribution of η_t^n.

Main Results

Theorem 1 (Jara-Landim-T., 19+)

Assume that $n^2\varepsilon_n^4 \le C_0 g_d(n)$ *for some constant* C_0, *where*

$$g_d(n) \;=\; n \;, \log n \;, 1 \;, \qquad \textit{if } d=1, d=2, d\ge 3,$$

respectively. Then, for any $T>0$, *there exists a constant* $C_1 = C_1(T, v_0, C_0)$ *such that for any* $0 \le t \le T$,

$$H(\mu_t^n|\nu_t^n) \;\le\; C_1 n^{d-2} g_d(n) \;,$$

where $H(\mu_t^n|\nu_t^n) = \int \frac{d\mu_t^n}{d\nu_t^n} \log \frac{d\mu_t^n}{d\nu_t^n} d\nu_t^n$ *(relative entropy).*

Corollary 2 (Jara-Landim-T., 19+)

Assume that $n^2\varepsilon_n^4 \le C_0 g_d(n)$ *and* $\varepsilon_n^2 n^2 g_d(n)^{-1} \to \infty$*. For any* $t \ge 0$ *and any smooth function* $G : \mathbb{T}^d \to \mathbb{R}$*,*

$$\lim_{n\to\infty} \mathbb{E}^n \left[\left| \frac{1}{\varepsilon_n n^d} \sum_{x \in \mathbb{T}_n^d} G(x/n)[\eta_t^n(x) - \alpha_0] - \int_{\mathbb{T}^d} G(x) v(t,x) dx \right| \right] = 0 ,$$

Remarks

- Initial distribution: The assumption $\eta_0^n \overset{\mathrm{d}}{=} \nu^n(\alpha_0 + \varepsilon_n v_0)$ can be replaced with the entropy bound at time 0.
- $\sigma'(\alpha_0) = 0$: In the case of general $\alpha \in (0,1)$, introducing the Galilean transformation $\alpha + \varepsilon_n v(t, x - \varepsilon_n^{-1}\sigma'(\alpha)\mathbf{m}t)$, we can obtain a similar result.
- In 1D case, $\varepsilon_n^2 n^2 g_d(n)^{-1} \to \infty \Leftrightarrow \varepsilon_n n^{1/2} \to \infty$. Since in the critical case ($\varepsilon_n = n^{-1/2}$) one can observe the Gaussian fluctuation, this condition seems unavoidable.

3. Sketch of the Proof

- Following [Jara-Menezes, 19+], we shall compute the entropy production. Let $H_t^n = H(\mu_t^n|\nu_t^n)$. Then, we have

$$\frac{d}{dt} H_t^n \;\le\; -n^2 D(g_t^n, L_n^S, \nu_t^n) + \int \left\{ L_n^{*,\nu_t^n} \mathbf{1} - \partial_t \log \nu_t^n \right\} d\mu_t^n \;,$$

  where g_t^n represents $d\mu_t^n / d\nu_t^n$, L_n^{*,ν_t^n} the adjoint of L_n in $L^2(\nu_t^n)$, $D(g_t^n, L_n^S, \nu_t^n)$ the Dirichlet form

$$D(g_t^n, L_n^S, \nu_t^n) \;=\; \sum_{x \in \mathbb{T}_n^d} \sum_{j=1}^{d} \int \left\{ \sqrt{g_t^n(\eta^{x,x+e_j})} - \sqrt{g_t^n(\eta)} \right\}^2 \nu_t^n(d\eta) \;.$$

- We need to compute the integrand $L_n^{*,\nu_t^n} \mathbf{1} - \partial_t \log \nu_t^n$ explicitly. Indeed, it can be expressed in terms of the "Fourier coefficients" of $g_{j,p}$ (but quite messy...).

Computation of $L_n^{*,\nu_t^n}\mathbf{1} - \partial_t \log \nu_t^n$

- Since $\nu_t^n = \nu^n(u_t^n)$ is just the Bernoulli measure with mean $u_t^n = \alpha_0 + \varepsilon_n v_t$, one can easily obtain

$$\partial_t \log \nu_t^n \ = \ \varepsilon_n \sum_{x\in\mathbb{T}_n^d} (\partial_t v)(t,x)\omega_x \ ,$$

  where

$$\omega_x = \frac{\eta_x - u_t^n(x)}{u_t^n(x)(1-u_t^n(x))} \ , \quad x \in \mathbb{T}_n^d \ .$$

- Recall that $L_n = n^2 L_n^S + \varepsilon_n^{-1} n L_n^A$. So the adjoint operator $L_n^{*,\nu_t^n}\mathbf{1}$ can be written as

$$L_n^{*,\nu_t^n}\mathbf{1} \ = \ n^2 L_n^{*,S,\nu_t^n}\mathbf{1} + \varepsilon_n^{-1} n L_n^{*,A,\nu_t^n}\mathbf{1} \ .$$

- After long computations, one can obtain

$$L_n^{*,\nu_t^n}\mathbf{1} \;=\; \sum_{j,x} K_j^n(t,x)\omega_x + \sum_{j,x}\sum_{A:|A|\geq 2} H_j^n(t,x,A)\omega(A+x)\ ,$$

for some $K_j^n(t,x)$, $H_j^n(t,x,A)$, where

$$\omega(B) \;=\; \prod_{x\in B}\omega_x\ , \quad B\subset \mathbb{T}_n^d\ .$$

- $K_j^n(t,x)$ can be computed as

$$\begin{aligned} K_j^n(t,x) \;&=\; n^2\left\{E_{\nu_t^n}\left[j_{x-e_j,x}\right] - E_{\nu_t^n}\left[j_{x,x+e_j}\right]\right\} + \varepsilon_n^{-1} n\widetilde{I}_j(t,x-e_j) \\ &\sim\; \nabla_n\cdot[D(u_t^n)\nabla_n u_t^n] + \varepsilon_n^{-1}\nabla_n\cdot[\sigma(u_t^n)\mathbf{m}]\ . \end{aligned}$$

- Due to the choice of the reference density u_t^n, the degree one terms do not vanish. However, we can show

$$\sup_{j,x} \left| \varepsilon_n \partial_t v(t, x/n) - K_j^n(t,x) \right| \leq C(T) \left(\varepsilon_n^2 + 1/n \right) .$$

  for any $0 \leq t \leq T, n \in \mathbb{N}$.
- The entropy inequality tells us that

$$\int f d\mu_t^n \leq \frac{1}{\gamma} \left[H(\mu_t^n | \nu_t^n) + \log \int e^{\gamma f} d\nu_t^n \right] ,$$

  for any function $f : \{0,1\}^{\mathbb{T}_n^d} \to \mathbb{R}$ and any $\gamma > 0$.
- Control of $\log \int \exp \left\{ \sum_{j,x} K_j^n(t,x) \omega_x \right\} d\nu_t^n$ is easy.
- The second and higher degree terms can be controlled by following Jara-Menezes's argument.

Thank you for your attention.