Stochastic Lagrangian path for Leray solutions of 3D Navier-Stokes equations

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(A joint work with Guohuan Zhao)

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Abstract

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$$\partial_t \mathbf{u} = \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p, \quad \mathrm{div}\, \mathbf{u} \equiv 0, \ \mathbf{u}_0 = \varphi.$$

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- For Lebesgue almost all (s, x), the solution $X^n_{s,\cdot}(x)$ associated with the mollifying velocity field $\mathbf{u}_n$ weakly converges to $X_{s,\cdot}(x)$.

Introduction

- Let $d \geqslant 2$ and consider the following Navier-Stokes equation:

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$$\|\mathbf{u}\|_{L^\infty([0,T];L^2(\mathbb{R}^d))} + \|\nabla \mathbf{u}\|_{L^2([0,T];L^2(\mathbb{R}^d))} < \infty, \quad \forall T > 0.$$

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- Existence and smoothness of Leray solutions are open problems!

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- Question: For any Leray solution **u**, is it possible to construct the stochastic Lagrangian particle trajectory associated with **u**?
- More precisely, for each starting point x, is there a unique solution to the following SDE?

$$\mathrm{d}X_t = \mathbf{u}(t, X_t)\mathrm{d}t + \sqrt{2\nu}\mathrm{d}W_t, \ X_0 = x, \tag{1.1}$$

where W is a d-dimensional standard Brownian motion on some probability space $(\Omega, \mathscr{F}, \mathbf{P})$.

Introduction

- If **u** is smooth in x, then by Constantin and Iyer's representation (2008, CPAM), **u** can be reconstructed from $X_t(x)$ as follows:

$$\mathbf{u}(t,x) = \mathcal{P}\mathbb{E}(\nabla^{\mathrm{t}} X_t^{-1}(x) \cdot \varphi(X_t^{-1}(x))),$$

  where $\mathcal{P}$ is the Leray projection and $X_t^{-1}(x)$ is the inverse of stochastic flow $x \mapsto X_t(x)$, and ∇^{t} is the transpose of a Jacobian matrix.

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- Krylov and Röckner (2005, PTRF) showed the existence-uniqueness of strong solutions to SDE (1.1) under the following assumption

$$\mathbf{u} \in \cap_{T>0} L^q([0,T]; L^p(\mathbb{R}^d)), \quad p, q \geqslant 2, \quad \frac{d}{p} + \frac{2}{q} < 1. \tag{1.2}$$

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- The unique solution $X_t(x)$ is weakly differentiable in x and satisfies (see Fedrizzi-Flandoli (2013), Z. (2013), (2016)):

$$\sup_{x \in \mathbb{R}^d} \mathbf{E} \left(\sup_{t \in [0,T]} |\nabla X_t(x)|^p \right) < \infty, \quad \forall p \geqslant 1, \quad T > 0.$$

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but only satisfies

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- Deterministic Lagrangian particle trajectories associated with $\mathbf{u}$ have been studied very well (see Robinson, Rodrigo and Sadowski's book).

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where $b(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable vector field.

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- Under the following assumption

$$b \in \cap_{T>0} L^q([0,T]; L^p(\mathbb{R}^d)) =: L^q_{loc}(L^p),\quad p,q \geqslant 2,\quad \frac{d}{p} + \frac{2}{q} < 2,$$

is there a semimartingale solution of SDE (1.5)? That is,

$$X_{s,t} = x + \int_s^t b(r, X_{s,r})\mathrm{d}r + \sqrt{2}(W_t - W_s),\ \ \forall t \geqslant s\ ?? \tag{1.6}$$

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- Z.-Zhao (2017): Weak well-posedness in the class of Dirichlet processes when $b \in H^{-\alpha,p}$ with $\alpha \in (0, \frac{1}{2}]$ and $p \in (\frac{d}{1-\alpha}, \infty)$.
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- Critical case ($\frac{d}{p} + \frac{2}{q} = 1$ with $p, q \in (2, \infty)$):

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- Kinzebulatov and Semenov (2017) showed the existence of weak solutions for each starting point $x \in \mathbb{R}^d$ when $b \in L^d(\mathbb{R}^d)$ is *time-independent*, but the uniqueness is left open.

A counter-example

- Consider the following concrete SDE:

$$X_t = -c \int_0^t X_s |X_s|^{-2} \mathrm{d}s + W_t, \quad c \in \mathbb{R}. \tag{1.7}$$

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- If $c \geqslant d$, Kinzebulatov and Semenov showed that the above SDE does not allow a solution. If $c < c_d$, where $c_d \in (0, d)$ is some constant only depending on d, they proved that there exists a weak solution to the above SDE by utilizing the analytic construction of the semigroup $\mathrm{e}^{-t(\Delta + b\cdot\nabla)}$.

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- Intuitively, if $c \geqslant d$, then the centripetal force is so strong such that the particle can not escape from the origin immediately so that even though a random perturbation is added, there is no solution for SDE (1.7).

Main result

- Let $\mathscr{D}$ be the space of all smooth functions with compact supports and $\mathscr{D}'$ the dual space of $\mathscr{D}$, which is also called distribution space. The duality between $\mathscr{D}'$ and $\mathscr{D}$ is denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. In particular, if $f(t,x)$ and $g(t,x)$ are two real functions in $\mathbb{R} \times \mathbb{R}^d$, then

$$\langle\!\langle f, g \rangle\!\rangle = \int_{\mathbb{R}} \langle f(t), g(t) \rangle \mathrm{d}t \ \text{ with } \ \langle f(t), g(t) \rangle := \int_{\mathbb{R}^d} f(t,x) g(t,x) \mathrm{d}x.$$

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- For two distributions $f,g\in\mathscr{D}'$, one says that $f\leqslant g$ if for any non-negative $\varphi\in\mathscr{D}$,

$$\langle\langle f,\varphi\rangle\rangle\leqslant\langle\langle g,\varphi\rangle\rangle.$$

- For $\alpha \in \mathbb{R}$ and $p \in [1, \infty]$, let $H^{\alpha,p}$ be the usual Bessel potential space with norm:

$$\|f\|_{\alpha,p} := \|(\mathbb{I} - \Delta)^{\alpha/2} f\|_p = \left(\int_{\mathbb{R}^d} |(\mathbb{I} - \Delta)^{\alpha/2} f(x)|^p \mathrm{d}x \right)^{1/p}.$$

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- For $\alpha \in \mathbb{R}$ and $p, q \in [1, \infty]$, let $\mathbb{H}^{\alpha,p}_q := L^q(\mathbb{R}; H^{\alpha,p})$ be the space of spatial-time functions with norm

$$\|f\|_{\alpha,p;q} := \left(\int_{\mathbb{R}} \|f(t, \cdot)\|^q_{\alpha,p} \mathrm{d}t \right)^{1/q}.$$

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- Fix $r > 0$. Let $\widetilde{\mathbb{H}}_q^{\alpha,p}$ be the Banach space of all functions $f \in \mathbb{H}_{q,loc}^{\alpha,p}$ with

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- For $p' > p, q' > q$, we have $\widetilde{\mathbb{H}}_{q'}^{\alpha,p'} \subset \widetilde{\mathbb{H}}_q^{\alpha,p}$ $(\mathbb{H}_{q'}^{\alpha,p'} \not\subset \mathbb{H}_q^{\alpha,p})$.

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- All the probability measures over $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ is denoted by $\mathscr{P}(\mathbb{C})$.
- Let ω_t be the canonical process over $\mathbb{C}$. For $t \geqslant 0$, let $\mathcal{B}_t := \mathcal{B}_t(\mathbb{C})$ be the natural filtration generated by $\{\omega_s : s \leqslant t\}$.

Definition 1

For given $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we call a probability measure $\mathbb{P}_{s,x} \in \mathscr{P}(\mathbb{C})$ a martingale solution of SDE (1.5) with starting point (s, x) if

(i) $\mathbb{P}_{s,x}(\omega_t = x, t \leqslant s) = 1$, and for each $t > s$,

$$\mathbb{E}^{\mathbb{P}_{s,x}}\left(\int_s^t |b(r, \omega_r)| \mathrm{d}r\right) < \infty.$$

(ii) For all $f \in C_c^2(\mathbb{R}^d)$, M_t^f is a $\mathcal{B}_t$-martingale under $\mathbb{P}_{s,x}$, where

$$M_t^f(\omega) := f(\omega_t) - f(x) - \int_s^t \mathscr{L}_r^b f(\omega_r) \mathrm{d}r, \quad t \geqslant s.$$

All the martingale solution $\mathbb{P}_{s,x}$ with starting point (s, x) and drift b is denoted by $\mathscr{M}_{s,x}^b$.

- Let $\mathbb{P}_{s,x} \in \mathscr{M}^b_{s,x}$. By Lévy's characterization for Brownian motion, one sees that

$$W_t := \frac{\sqrt{2}}{2}\left(\omega_t - \omega_s - \int_s^t b(r,\omega_r)\mathrm{d}r\right), \quad t \geqslant s,$$

is a d-dimensional standard Browian motion under $\mathbb{P}_{s,x}$, so that

$$\omega_t = x + \int_s^t b(r,\omega_r)\mathrm{d}r + \sqrt{2}W_t, \quad t \geqslant s.$$

In other words, $(\mathbb{C}, \mathscr{B}(\mathbb{C}), \mathbb{P}_{s,x}, \omega_t, W_t)$ is a weak solution of SDE (1.5).

Theorem 2

Suppose that for some $p_i, q_i \in [2,\infty)$ *with* $\frac{d}{p_i}+\frac{2}{q_i}<2$, $i=1,2$,

$$|||b|||_{0,p_1;q_1}+|||(\mathrm{div}\, b)^-|||_{0,p_2;q_2}<\infty. \tag{2.1}$$

For each $(s,x)\in\mathbb{R}_+\times\mathbb{R}^d$, *there exists at least one martingale solution* $\mathbb{P}_{s,x}\in\mathscr{M}^b_{s,x}$, *which satisfies the following Krylov's type estimate: for any* $\alpha\in[0,1]$ *and* $p,q\in(1,\infty)$ *with* $\frac{d}{p}+\frac{2}{q}<2-\alpha$, *there exist* $\theta=\theta(\alpha,p,q)>0$ *and a constant* $C>0$ *such that for all* $s\leqslant t_0<t_1<\infty$ *with* $t_1-t_0\leqslant 1$ *and* $f\in C_c^\infty(\mathbb{R}^{d+1})$,

$$\mathbb{E}^{\mathbb{P}_{s,x}}\left(\int_{t_0}^{t_1} f(t,\omega_t)\mathrm{d}t\,\middle|\,\mathcal{B}_{t_0}\right)\leqslant C(t_1-t_0)^\theta|||f|||_{-\alpha,p;q}. \tag{2.2}$$

Ctd.

Moreover, we have the following conclusions

(i) (Weak uniqueness) For any mollifying approximation b_n of b, there is a Lebesgue-null set $\mathcal{N} \subset \mathbb{R}_+ \times \mathbb{R}^d$ such that for all $(s,x) \in \mathcal{N}^c$,

$$\mathbb{P}^n_{s,x} \text{ weakly converges to } \mathbb{P}_{s,x} \in \mathscr{M}^b_{s,x}, \text{ where } \mathbb{P}^n_{s,x} \in \mathscr{M}^{b_n}_{s,x}.$$

(ii) (Almost surely Markov property) For each $(s,x) \in \mathcal{N}^c$, there is a Lebesgue null set $I_{s,x} \subset [s,\infty)$ such that for all $t_0 \in (s,\infty) \setminus I_{s,x}$, any $t_1 > t_0$ and $f \in C_c(\mathbb{R}^d)$,

$$\mathbb{E}^{\mathbb{P}_{s,x}}(f(\omega_{t_1})|\mathcal{B}_{t_0}) = \mathbb{E}^{\mathbb{P}_{t_0,\omega_{t_0}}}(f(\omega_{t_1})), \quad \mathbb{P}_{s,x} - a.s.$$

(iii) (L^p-semigroup) Let $\mathcal{T}_{s,t}f(x) := \mathbb{E}^{\mathbb{P}_{s,x}} f(\omega_t)$. For any $p \geqslant 1$ and $T > 0$, there is a constant $C > 0$ such that for Lebesgue almost all $0 \leqslant s < t \leqslant T$ and $f \in L^p(\mathbb{R}^d)$,

$$\|\mathcal{T}_{s,t}f\|_p \leqslant C\|f\|_p. \tag{2.3}$$

Application to stochastic Lagrangian particle path of 3D-NSEs

- If $(\mathrm{div} b)^- \equiv 0$, then $\|\mathcal{T}_{s,t} f\|_1 \leqslant \|f\|_1$ in (2.3). If $\mathrm{div} b \equiv 0$, then for any nonnegative $f \in L^1(\mathbb{R}^d)$, $\|\mathcal{T}_{s,t} f\|_1 = \|f\|_1$. By (1.4), we can apply the above theorem to the Leray solution of 3D-NSEs.

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- Under the assumptions

$$\nabla b \in \mathbb{L}^1_{loc}, \quad (\mathrm{div} b)^-, b/(1+|x|) \in \mathbb{L}^\infty,$$

the existence and uniqueness of almost everywhere stochastic flows are obtained in the framework of DiPerna-Lions' theory has been obtained in Zhang (2010). However, the existence of a solution is only shown for Lebesgue almost all starting point $x \in \mathbb{R}^d$.

Example

Let $d \geqslant 3$ and $\alpha < 3$. Define

$$b(x) := \sum_{z \in \mathbb{Z}^d} \gamma_z \frac{x - z}{|x - z|^\alpha} \phi(|x - z|),$$

where for some $M > 0$, $\gamma_z \in (0, M)$ is a constant and $\phi \in C_c^\infty(\mathbb{R}_+; [0, 1])$ with $\phi(r) = 1$ for $r \in [0, 1]$ and $\phi(r) = 0$ for $r > 2$. It is easy to see that (2.1) holds.

Idea of Proof

- We assume that for some $p_i, q_i \in [2, \infty)$ with $\frac{d}{p_i} + \frac{2}{q_i} < 2$, $i = 1, 2$,

$$\kappa := |||b|||_{p_1;q_1} + |||(\mathrm{div} b)^-|||_{p_2;q_2} < \infty.$$

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- Let $b_n(t, x) = b(t, \cdot) * \rho_n(x)$ be the mollifying approximation of $b(t, \cdot)$. It is easy to check that
$$\sup_n \left(|||b_n|||_{p_1;q_1} + |||(\mathrm{div} b_n)^-|||_{p_2;q_2} \right) \leqslant C\kappa,$$
and
$$b_n \in L^{q_1}_{loc}(\mathbb{R}_+; C^\infty_b(\mathbb{R}^d)).$$

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- For $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, consider the following SDE:

$$\mathrm{d}X^n_{s,t} = b_n(t, X^n_{s,t})\mathrm{d}t + \sqrt{2}\mathrm{d}W_t, \quad X^n_{s,s} = x, \quad t \geqslant s,$$

where W is a d-dimensional standard Brownian motion on some complete filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbf{P})$.

- We want to show that the law $\mathbb{P}^n_{s,x}$ of $X^n_{s,\cdot}(x)$ is tight in $\mathbb{C}$.

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- By the strong Markov property, it suffices to show

$$\sup_n\sup_{s,x}\mathbb{E}\int_0^\delta b_n(s+t,X^n_{s,t}(x))\mathrm{d}t\leqslant c_\delta\to 0.$$

Lemma 3

For any $\alpha \in [0,1]$ *and* $p, q \in (1,\infty)$ *with* $\frac{d}{p} + \frac{2}{q} < 2 - \alpha$, *there are constants* $\theta = \theta(\alpha, p, q) > 0$ *and* $C > 0$ *depending on* $\kappa, d, \alpha, p, q, p_i, q_i$ *such that for any* $f \in C_c^\infty(\mathbb{R}^{d+1})$ *and* $0 \leqslant s \leqslant t_0 < t_1 < \infty$ *with* $t_1 - t_0 \leqslant 1$,

$$\sup_n \sup_{x\in\mathbb{R}^d} \mathbf{E}\left(\int_{t_0}^{t_1} f(t, X^n_{s,t}(x))\mathrm{d}t \middle| \mathscr{F}_{t_0}\right) \leqslant C(t_1 - t_0)^\theta |||f|||_{-\alpha,p;q}.$$

- Fix $0 \leqslant s \leqslant t_0 < t_1 < \infty$ with $t_1 - t_0 \leqslant 1$ and $f \in C_c^\infty(\mathbb{R}^{d+1})$. Let u_n be the smooth solution of the following backward PDE:

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- By Itô's formula we have

$$u_n(t_1, X^n_{s,t_1}) = u_n(t_0, X^n_{s,t_0}) - \int_{t_0}^{t_1} f(t, X^n_{s,t}) \mathrm{d}t + \sqrt{2} \int_{t_0}^{t_1} \nabla u_n(t, X^n_{s,t}) \mathrm{d}W_t.$$

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- By taking conditional expectation with respect to $\mathscr{F}_{t_0}$, we obtain

$$\mathbf{E}\left(\int_{t_0}^{t_1} f(t, X^n_{s,t}) \mathrm{d}t \middle| \mathscr{F}_{t_0} \right) = \mathbf{E}\Big(u_n(t_0, X^n_{s,t_0}) | \mathscr{F}_{t_0}\Big) \leqslant \|u_n(t_0)\|_\infty.$$

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- The key point is to show that

$$\|u_n(t_0)\|_\infty \leqslant C|||f1_{[t_0,t_1]}|||_{-\alpha,p;q'} \leqslant C(t_1 - t_0)^{1-\frac{q'}{q}}|||f|||_{-\alpha,p;q},$$

where $q' < q$ so that $\frac{d}{p} + \frac{2}{q'} < 2 - \alpha$.

Maximal principle by De-Giorgi's argument

- Let $\mathscr{V} := \mathbb{L}^2_\infty \cap \mathbb{H}^{1,2}_2$, $\mathscr{V}_{loc} := \mathbb{L}^2_{\infty,loc} \cap \mathbb{H}^{1,2}_{2,loc}$.

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- We assume

$$b \in \mathbb{L}^2_{2,loc}, \quad f \in \mathscr{D}',$$

- Consider the following PDE in $\mathbb{R}^{d+1}$:

$$\partial_t u = \Delta u + b \cdot \nabla u + f. \tag{4.1}$$

Definition 4

A function $u \in \mathscr{V}_{loc} \cap \mathbb{L}^\infty_{loc}$ is called a weak solution of PDE (4.1) if for any nonnegative smooth function $\varphi \in C^\infty_c(\mathbb{R}^{d+1})$,

$$\langle\!\langle \partial_t u, \varphi \rangle\!\rangle = -\langle\!\langle \nabla u, \nabla \varphi \rangle\!\rangle + \langle\!\langle b \cdot \nabla u, \varphi \rangle\!\rangle + \langle\!\langle f, \varphi \rangle\!\rangle.$$

Theorem 5 (Global maximum estimate)

Suppose that for some $\alpha_i \in [0,1]$ *and* $p_i, q_i \in (1,\infty)$ *with* $\frac{d}{p_i} + \frac{2}{q_i} < 2 - \alpha_i$, $i = 1,2,3$,

$$b \in \widetilde{\mathbb{H}}^{-\alpha_1,p_1}_{q_1}, \quad -\mathrm{div} b \leqslant \Theta_b \in \widetilde{\mathbb{H}}^{-\alpha_2,p_2}_{q_2}, \quad f \in \widetilde{\mathbb{H}}^{-\alpha_3,p_3}_{q_3}. \tag{4.2}$$

Let $u \in \mathscr{V}_{loc} \cap \mathbb{L}^\infty_{loc}$ *be a weak solution of PDE* (4.1) *with initial value* $u(0) = 0$. *For any* $T > 0$, *there exists a constant* $C > 0$ *depending only on* T, d, α_i, p_i, q_i *and the quantity*

$$\kappa := |||b|||_{-\alpha_1,p_1;q_1} + |||\Theta_b|||_{-\alpha_1,p_1;q_1}$$

such that

$$\|u\|_{L^\infty([0,T]\times\mathbb{R}^d)} + |||u1_{[0,T]}|||_{\mathscr{V}} \leqslant C|||f1_{[0,T]}|||_{-\alpha_3,p_3;q_3}.$$

- When $f \equiv 0$, under (4.2) with $\alpha_i = 0$, the local maximum principle is proved by Nazarov and Ural'tseva (2012) by using Moser's iteration.

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- In elliptic case with $b = 0$ and $f \in L^p(\mathbb{R}^d)$ for $p > \frac{d}{2}$, Han-Lin (2011) show the same maximum principle by De-Giorgi and Moser's iterations.

Thank you for your kind attention!