Well-posedness of renormalized solutions for a stochastic p-Laplace equation with L^1-initial data

Aleksandra Zimmermann
University of Duisburg-Essen and University of Rostock
(joint work with Niklas Sapountzoglou)

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The stochastic p-Laplace problem

$T > 0$, $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain, $Q_T := (0, T) \times D$.

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- $u_0 \in L^1(\Omega \times D)$ is $\mathcal{F}_0$-measurable.
- The right-hand side is an integral in the sense of Itô with respect to $(\beta_t)_{t \geq 0}$, $\Phi \in L^2(\Omega \times Q_T)$ *progressively measurable*.

Motivation

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- Randomness can be introduced as random external force by adding a stochastic integral on the right-hand side of the equation and by considering random initial values.
- The initial values may have poor regularity with respect to both variables.

Existence and uniqueness for L^2- initial data

A unique strong solution can be obtained using classical monotonicity methods for SPDEs [Pardoux;1975], [Krylov, Rozovskii; 1983], [Liu, Röckner; 2015],...

Theorem

Let $u_0 \in L^2(\Omega \times D)$ be $\mathcal{F}_0$-measurable. There exists a unique, $\mathcal{F}_t$-adapted, square-integrable stochastic process $u : \Omega \times [0, T] \to L^2(D)$ with a.s. continuous paths such that $u(0, \cdot) = u_0$, $u \in L^p(\Omega; L^p(0, T; V))$ and

$$u(t) - u_0 - \int_0^t \operatorname{div}\left(|\nabla u(s)|^{p-2}\nabla u(s)\right) ds = \int_0^t \Phi(s)\, d\beta_s$$

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- For $p \geq \frac{2d}{d+2}$, $V = W_0^{1,p}(D)$.
- For $1 < p < \frac{2d}{d+2}$, $V = W_0^{1,p}(D) \cap L^2(D)$.

What about merely integrable initial data?

- The Itô formula yields the energy estimate

$$\frac{1}{2}\|u(t)\|^2_{L^2(D)} - \frac{1}{2}\|u_0\|^2_{L^2(D)} + \int_0^t \|\nabla u(s)\|^p_{L^p(D)}\, ds$$
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What about merely integrable initial data?

- For $\mathcal{F}_0$-measurable $u_0 \in L^2(\Omega \times D)$, the Itô formula yields the energy estimate

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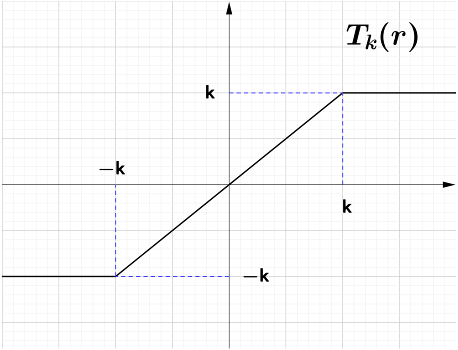
- Renormalized solutions: Introduced by DiPerna and Lions 1989 in the study of the Boltzmann equation.
- Main idea: Nonlinear change of unknown $v = S(u)$, where S is chosen in order to remove infinite quantities.

Renormalization and truncation

For $S \in \mathcal{C}^2(\mathbb{R})$ such that $\mathrm{supp}(S') \subset [-M, M]$ for some $M > 0$, S is constant outside $[-M, M]$. Thus, $S(u)(t) = S(T_k(u))(t)$ for all $k \geq M$.



For $u \in W_0^{1,p}(D)$, from the chain rule for Sobolev functions it follows that

$$S'(u)\nabla u = S'(u)\chi_{\{|u| \leq k\}}\nabla u = S'(u)T_k'(u)\nabla u = S'(u)\nabla T_k(u).$$

Derivation of the renormalized equation

Let u be a strong solution of the stochastic p-Laplace problem with $\mathcal{F}_0$-measurable initial value $u_0 \in L^1(\Omega \times D)$, i.e.,

$$u(t) - u_0 - \int_0^t \operatorname{div}\left(|\nabla u(s)|^{p-2}\nabla u(s)\right) ds = \int_0^t \Phi(s)\, d\beta_s.$$

For $S \in \mathcal{C}^2(\mathbb{R})$ with bounded derivatives, the Itô formula yields

$$S(u)(t) - S(u_0) - \int_0^t \operatorname{div}\left[S'(u)|\nabla u|^{p-2}\nabla u\right] ds$$
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$$dS(u)(t) - \operatorname{div}\left[S'(u)|\nabla u|^{p-2}\nabla u\right] dt + S''(u)\left[|\nabla u|^p - \frac{1}{2}|\Phi|^2\right] dt$$
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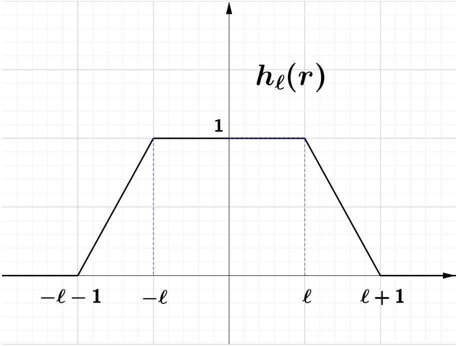
Consistency of renormalized solutions

Let u be a renormalized solution with $\nabla u \in L^p(\Omega \times Q_T)^d$, $\ell > 0$.



Choosing

$$S_\ell(u) = \int_0^u h_\ell(r)\, dr$$

in the renormalized equation $(R2)$, for $\ell \to \infty$, it follows that u is a strong solution.

L^1-contraction principle

Theorem

Let u, v be renormalized solutions with $\mathcal{F}_0$-measurable initial values $u_0 \in L^1(\Omega \times D)$ and $v_0 \in L^1(\Omega \times D)$, respectively. Then,

$$\sup_{t \in [0,T]} \|u(t) - v(t)\|_{L^1(D)} \leq \|u_0 - v_0\|_{L^1(D)}$$

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Remark: As a consequence, renormalized solutions are unique.

Idea of the proof:

Let u, v be renormalized solutions with initial values u_0, v_0. Subtracting the renormalized equation $(R2)$ for v from the one for u we obtain

$$d[S(u) - S(v)] - \operatorname{div}[S'(u)|\nabla u|^{p-2}\nabla u - S'(v)|\nabla v|^{p-2}\nabla v]\,dt$$
$$+ S''(u)|\nabla u|^p - S''(v)|\nabla v|^p\,dt - \frac{1}{2}\Phi^2(S''(u) - S''(v))\,dt$$
$$= \Phi(S'(u) - S'(v))\,d\beta_t.$$

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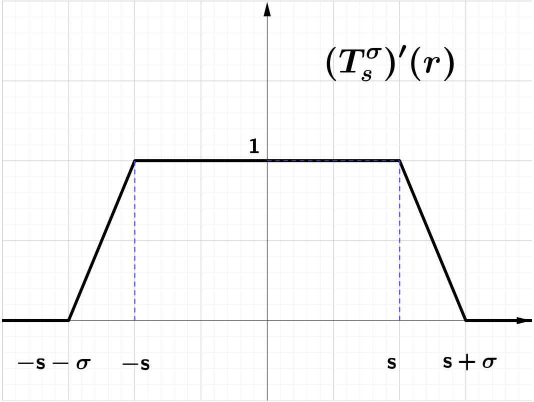
$$d|u-v| = \text{something} \quad \leq 0$$

Formally, we choose $S(r) = r$ and use the Itô formula for the absolute value.

Idea of the proof [Blanchard, Murat, Redwane; 2001]:

For $s, \sigma > 0$, we choose $S(r) = T_s^\sigma(r) := \int_0^r (T_s^\sigma)'(\tau)\, d\tau$ with



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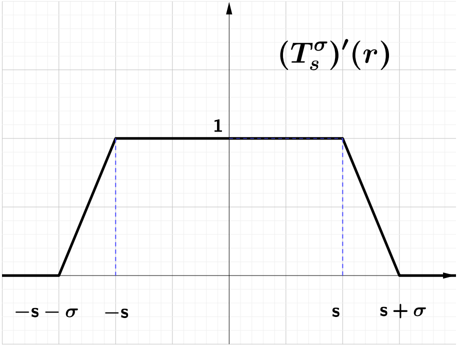

Itô formula for an approximation of the absolute value:

$$G_k(T_s^\sigma(u) - T_s^\sigma(v)) = \frac{1}{k} \int_0^{T_s^\sigma(u) - T_s^\sigma(v)} T_k(\tau)\, d\tau$$

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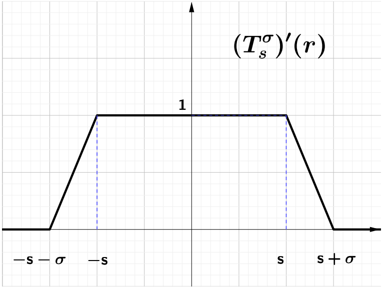

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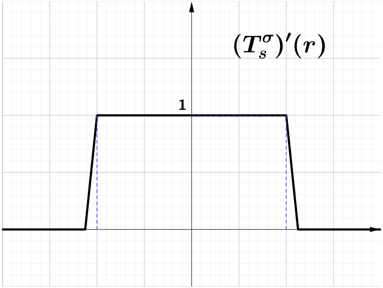
$$G_k(T_s^\sigma(u) - T_s^\sigma(v)) = \frac{1}{k} \int_0^{T_s^\sigma(u) - T_s^\sigma(v)} T_k(\tau)\, d\tau.$$

Thanks to the energy dissipation condition $(R3)$ we can pass to the limit with $\sigma \to 0$

Idea of the proof [Blanchard, Murat, Redwane; 2001]:

For $s, \sigma > 0$, we choose $S(r) = T_s^\sigma(r) := \int_0^r (T_s^\sigma)'(\tau)\, d\tau$ with



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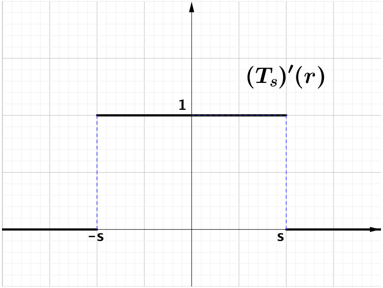

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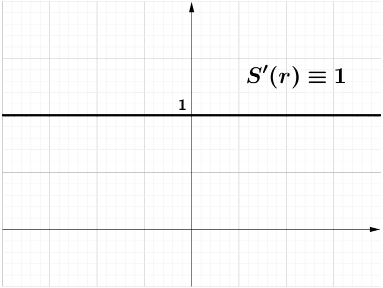

Itô formula for an approximation of the absolute value:

$$G_k(u - v) = \frac{1}{k} \int_0^{u-v} T_k(\tau)\, d\tau.$$

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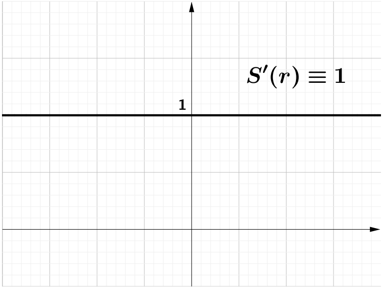

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Existence of renormalized solutions

Theorem (Existence)

For any $\mathcal{F}_0$-measurable initial value $u_0 \in L^1(\Omega \times D)$, there exists a renormalized solution to the stochastic p-Laplace problem.

L^1-convergence of approximate solutions

- Let $(u_0^n)_n \subset L^2(\Omega \times D)$ be an $\mathcal{F}_0$-measurable sequence such that $u_0^n \to u_0$ in $L^1(\Omega \times D)$ for $n \to \infty$.

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- If u_n, u_m are strong solutions to the stochastic p-Laplace problem with initial values u_0^n, u_0^m respectively, from the L^1-contraction principle we get

$$\sup_{t\in[0,T]} \|u_n(t) - u_m(t)\|_{L^1(D)} \leq \|u_0^n - u_0^m\|_{L^1(D)}$$

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$$\lim_{n \to \infty} u_n = u$$

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- We claim that this function u is a renormalized solution with initial value u_0.

Passage to the limit in the renormalized equation

For all $S \in \mathcal{C}^2(\mathbb{R})$ such that S' has compact support, the sequence of approximate solutions $(u_n) \subset L^1(\Omega \times Q_T)$, satisfies

$$S(u_n)(t) - S(u_0^n) - \int_0^t \operatorname{div}\left[S'(u_n)|\nabla u_n|^{p-2}\nabla u_n\right] ds$$

$$+ \int_0^t S''(u_n)\left[|\nabla u_n|^p - \frac{1}{2}\Phi^2\right] ds = \int_0^t S'(u_n)\Phi \, d\beta_s$$

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For the passage to the limit we need convergence of

$$\operatorname{div}\left[S'(u_n)|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n)\right] + S''(u_n)|\nabla T_k(u_n)|^p$$

in $L^{p'}(0, t; W^{-1,p'}(D) + L^1(D))$, a.s. in Ω for $n \to \infty$ and any $k > 0$.

Strong convergence of the truncated gradients

...the method of [Blanchard; 1993] is to show that $(u_n)_{n\in\mathbb{N}}$ satisfies

$$\lim_{n,m\to\infty} \mathbb{E}\int_{Q_T} \left(|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u_m)|^{p-2}\nabla T_k(u_m)\right)\cdot$$

$$\cdot\nabla(T_k(u_n) - T_k(u_m))\, d(s,x) = 0.$$

As a consequence,

$$\lim_{n\to\infty} \nabla T_k(u_n) = \nabla T_k(u)$$

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We need to find an equation for $Z(u_n-u_m)H(u_n)$ where H, Z are appropriate nonlinear test functions.

Itô product rule for approximate solutions

For $m, n \in \mathbb{N}$, u_0^n, $u_0^m \in L^2(\Omega \times D)$ $\mathcal{F}_0$-measurable, $t \in [0, T]$

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Since $\ll u_n - u_m, u_n \gg_t = 0$, we obtain

$$(u_n - u_m, u_n)_2\,(t) - (u_0^n - u_0^m, u_0^n)_2 + \int_0^t \langle \Delta_p(u_n) - \Delta_p(u_m), u_n \rangle\, ds$$

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&(Z(u_n - u_m), H(u_n))_2(t) = (Z(u_0^n - u_0^m), H(u_0^n))_2 \\
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&+ \int_0^t \langle \Delta_p(u_n), H'(u_n) Z(u_n - u_m) \rangle \, ds \\
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Markov properties in L^1

- The unique renormalized solution $u = u(t, r, u_r)$ of the stochastic p-Laplace equation starting in u_r at time $r \in [0, T]$ is a time-homogeneous Markov process with transition semigroup

$$P_t(\varphi)(x) := \mathbb{E}\,\varphi(u(t, 0, x)), \ \ t \in [0, T]$$

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- $(P_t)_{t\in[0,T]}$ has the *e-property*: For any $\varphi : L^1(D) \to \mathbb{R}$ bounded and Lipschitz continuous, $x \in L^1(D)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

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- The distribution $(\mathbb{P}_x)_{x\in L^1(D)}$ of $u(\cdot, 0, x)$ is a Markov process on $\mathcal{C}([0, T]; L^1(D))$.

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Adding the first-order term

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- In this case, existence and uniqueness of strong solutions has been obtained by time discretization and stochastic compactness arguments [Vallet, Z.;2018+2019].
- General case: $F(u) \notin L^1(\Omega \times Q_T)$, however $F(T_k(u)) \in L^1(\Omega \times Q_T)$ for all $k > 0$.

Thank you for your attention.